Random Non-Crossing Plane Configurations: A Conditioned Galton-Watson Tree Approach

Nicolas Curien,[1] Igor Kortchemski[2]
[1]Département de Mathématiques et Applications, École Normale Supérieure, 45 rue d'Ulm, 75230 Paris Cedex 05, France; e-mail: nicolas.curien@ens.fr
[2]Laboratoire de mathématiques, UMR 8628 CNRS, Université Paris-Sud, 91405 Orsay Cedex, France; e-mail: igor.kortchemski@normalesup.org

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ABSTRACT: We study various models of random non-crossing configurations consisting of diagonals of convex polygons, and focus in particular on uniform dissections and non-crossing trees. For both these models, we prove convergence in distribution towards Aldous' Brownian triangulation of the disk. In the case of dissections, we also refine the study of the maximal vertex degree and validate a conjecture of Bernasconi, Panagiotou and Steger. Our main tool is the use of an underlying Galton-Watson tree structure. © 2012 Wiley Periodicals, Inc. Random Struct. Alg., 45, 236–260, 2014

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1. INTRODUCTION

Various models of non-crossing geometric configurations involving diagonals of a convex polygon in the plane have been studied both in geometry, probability theory and especially in enumerative combinatorics (see e.g. [10]). Three specific models of non-crossing configurations—-triangulations, dissections and non-crossing trees—have drawn particular attention. Let us first recall the definition of these models.

Let P_n be the convex polygon inscribed in the unit disk of the complex plane whose vertices are the n-th roots of unity. By definition, a *dissection* of P_n is the union of the sides of P_n and of a collection of diagonals that may intersect only at their endpoints. A *triangulation* is a dissection whose inner faces are all triangles. Finally, a *non-crossing tree*

Correspondence to: Igor Kortchemski

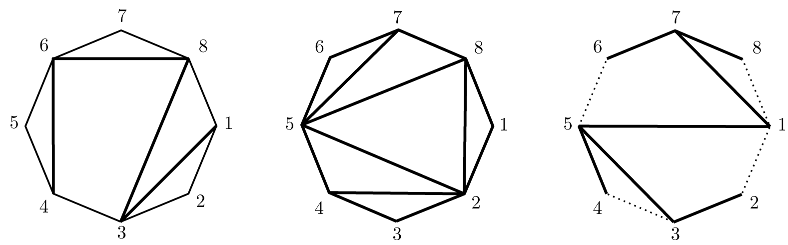

Fig. 1. A dissection, a triangulation and a non-crossing tree of the octogon.

of P_n is a tree drawn on the plane whose vertices are all vertices of P_n and whose edges are non-crossing line segments. See Fig. 1 for examples.

Graph theoretical properties of uniformly distributed triangulations have been recently investigated in combinatorics. For instance, the study of the asymptotic behavior of the maximal vertex degree has been initiated in [7] and pursued in [12]. Afterwards, the same random variable has been studied in the case of dissections [4].

We shall continue the study of graph-theoretical properties of large uniform dissections and in particular focus on the maximal vertex degree. Our method is based on finding and exploiting an underlying Galton-Watson tree structure. More precisely, we show that the *dual tree* associated with a uniformly distributed dissection of P_n is a critical Galton-Watson tree conditioned on having exactly $n-1$ *leaves*. This new conditioning of Galton-Watson trees has been studied recently in [14] (see also [22]) and is well adapted to the study of dissections, see [15]. In particular we are able to validate a conjecture contained in [4] concerning the asymptotic behavior of the maximal vertex degree in a uniform dissection (Theorem 4.7). Using the critical Galton-Watson tree conditioned to survive introduced in [13], we also give a simple probabilistic explanation of the fact that the inner degree of a given vertex in a large uniform dissection converges in distribution to the sum of two independent geometric variables (Proposition 4.6). We finally obtain new results about the asymptotic behavior of the maximal face degree in a uniformly distributed dissection.

As a by-product of our techniques, we give a very simple probabilistic approach to the following enumeration problem. Let $\mathcal{A}$ be a non-empty subset of $\{3, 4, 5, \ldots\}$ and $\mathbf{D}_n^{(\mathcal{A})}$ the set of all dissections of P_{n+1} whose face degrees all belong to the set $\mathcal{A}$. Theorem 3.7 gives an explicit asymptotic formula for $\#\mathbf{D}_n^{(\mathcal{A})}$ as $n \to \infty$ (for those values of n for which $\mathbf{D}_n^{(\mathcal{A})} \neq \varnothing$). In particular when $\mathcal{A}_0 = \{3, 4, 5, \ldots\}$, then $\mathbf{D}_{n-1} := \mathbf{D}_{n-1}^{(\mathcal{A}_0)}$ is the set of all dissections of P_n and

$$\#\mathbf{D}_{n-1} \underset{n\to\infty}{\sim} \frac{1}{4}\sqrt{\frac{99\sqrt{2}-140}{\pi}}\, n^{-3/2}(3+2\sqrt{2})^n.$$

This formula (Corollary 2.5) was originally derived by Flajolet & Noy [10] using very different techniques.

From a geometrical perspective, Aldous [2, 3] proposed to consider triangulations of P_n as closed subsets of the unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ rather than viewing them as graphs. He proved that large uniform triangulations of P_n converge in distribution (for the Hausdorff distance on compact subsets of the unit disk) towards a random compact subset. This limiting object is called *the Brownian triangulation* (see Fig. 2). This name comes from the fact that the Brownian triangulation can be constructed from the Brownian excursion

Fig. 2. A sample of the Brownian triangulation $\mathcal{B}$. [Color figure can be viewed in the online issue, which available at wileyonlinelibrary.com.]

as follows: Let $\mathbb{e} : [0, 1] \to \mathbb{R}$ be a normalized excursion of linear Brownian motion. For every $s, t \in [0, 1]$, we set $s \sim t$ if we have $\mathbb{e}(s) = \mathbb{e}(t) = \min_{[s\wedge t, s\vee t]} \mathbb{e}$. The Brownian triangulation is then defined as:

$$\mathcal{B} := \bigcup_{s\sim t} \left[e^{-2i\pi s}, e^{-2i\pi t} \right], \tag{1}$$

where $[x, y]$ stands for the Euclidean line segment joining two complex numbers x and y. It turns out that $\mathcal{B}$ is almost surely a closed set which is a continuous triangulation of the unit disk (in the sense that the complement of $\mathcal{B}$ in $\overline{\mathbb{D}}$ is a disjoint union of open Euclidean triangles whose vertices belong to the unit circle). Aldous also observed that the Hausdorff dimension of $\mathcal{B}$ is almost surely equal to $3/2$ (see [17]). Later, in the context of random maps, the Brownian triangulation has been studied by Le Gall & Paulin in [17] where it serves as a tool in the proof of the homeomorphism theorem for the Brownian map. See also [6, 15] for analogs of the Brownian triangulation.

However, neither large random uniform dissections, nor large uniform non-crossing trees have yet been studied from this geometrical point of view (Fig. 3).

In this work, we extend Aldous' theorem by showing that both large uniform dissections and large uniform non-crossing trees converge in distribution towards the Brownian triangulation (Theorem 3.1). The maybe surprising fact that large uniform dissections (which may have non-triangular faces) converge to a continuous triangulation stems from the fact that many diagonals degenerate in the limit. For both models, the key is to use a Galton-Watson tree structure, which was already described above in the case of dissections. In the case

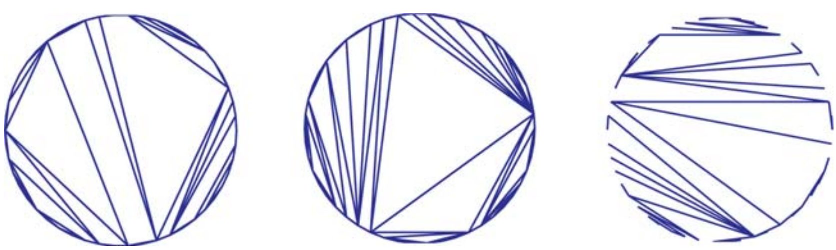

Fig. 3. Uniform dissection, triangulation and non-crossing tree of size 50. The same continuous model? [Color figure can be viewed in the online issue, which available at wileyonlinelibrary.com.]

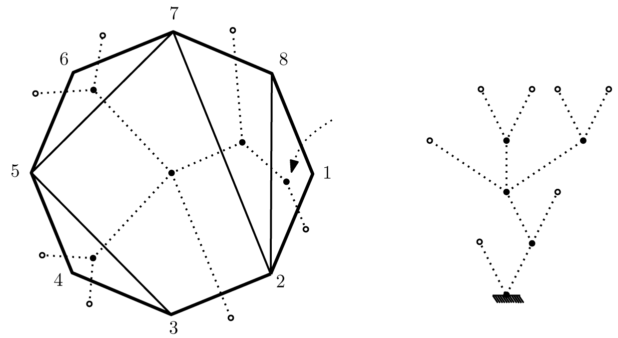

Fig. 4. The dual tree of a dissection of P_8, note that the tree has 7 leaves.

of non-crossing trees this structure has been identified by Marckert & Panholzer [20] who established that the *shape* of a uniform non-crossing tree of P_n is *almost* a Galton-Watson tree conditioned on having n vertices (see Theorem 2.7 below for a precise statement).

We also consider other random configurations of non-crossing diagonals of P_n such as non-crossing graphs, non-crossing partitions and non-crossing pair partitions, and prove the convergence towards the Brownian triangulation, again by using an appropriate underlying tree structure. We also show that a uniformly distributed dissection over $\mathbf{D}_n^{(\mathcal{A})}$ converges towards $\mathcal{B}$ as $n \to \infty$. The Brownian triangulation thus appears as a universal limit for random non-crossing configurations. This has interesting applications: For instance, let χ_n be a random non-crossing configuration on the vertices of P_n that converges in distribution towards $\mathcal{B}$ in the sense of the Hausdorff metric. Then the corresponding angle of the longest diagonal of χ_n, suitably normalized, converges in distribution towards the length of the longest chord of the Brownian triangulation with law

$$\frac{1}{\pi} \frac{3x-1}{x^2(1-x)^2\sqrt{1-2x}} \mathbf{1}_{\frac{1}{3} \leq x \leq \frac{1}{2}} \mathrm{d}x.$$

This has been shown in the particular case of triangulations in [3] (see also [7]).

The paper is organized as follows. In the second section we introduce the discrete models and explain the underlying Galton-Watson tree structures. The third section is devoted to the convergence of different random non-crossing configurations towards the Brownian triangulation and to applications. The final section contains the analysis of graph-theoretical properties of large uniform dissections, such as the maximal vertex or face degree.

2. DISSECTIONS, NON-CROSSING TREES AND GALTON-WATSON TREES

2.1. Dissections and Plane Trees

Throughout this work, for every integer $n \geq 3$, P_n stands for the regular polygon of the plane with n sides whose vertices are the n-th roots of unity.

Definition 2.1. A *dissection* $\mathcal{D}$ of the polygon P_n is the union of the sides of the polygon and of a collection of diagonals that may intersect only at their endpoints. A *face* f of $\mathcal{D}$ is a connected component of the complement of $\mathcal{D}$ inside P_n; its degree, denoted by $\deg(f)$, is the number of sides surrounding f. See Fig. 4 for an example.

Let $\mathbf{D}_n$ be the set of all dissections of P_{n+1}. Given a dissection $\mathcal{D} \in \mathbf{D}_n$, we construct a (rooted ordered) tree $\phi(\mathcal{D})$ as follows: Consider the dual graph of $\mathcal{D}$, obtained by placing a vertex inside each face of $\mathcal{D}$ and outside each side of the polygon P_{n+1} and by joining two vertices if the corresponding faces share a common edge, thus giving a connected graph without cycles. Then remove the dual edge intersecting the side of P_{n+1} which connects 1 to $e^{\frac{2i\pi}{n+1}}$. Finally, root the tree at the corner adjacent to the latter side (see Fig. 4).

The dual tree of a dissection is a plane tree (also known as rooted ordered tree in the literature). We briefly recall the formalism of plane trees which can be found in [16] for example. Let $\mathbb{N} = \{0, 1, \ldots\}$ be the set of nonnegative integers, $\mathbb{N}^* = \{1, \ldots\}$ and let $\mathcal{U}$ be the set of labels

$$\mathcal{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

where by convention $(\mathbb{N}^*)^0 = \{\varnothing\}$. An element of $\mathcal{U}$ is a sequence $u = u_1 \cdots u_m$ of positive integers, and we set $|u| = m$, which represents the "generation" of u. If $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_n$ belong to $\mathcal{U}$, we write $uv = u_1 \cdots u_m v_1 \cdots v_m$ for the concatenation of u and v. Finally, a *plane tree* τ is a finite or infinite subset of $\mathcal{U}$ such that:

1. $\varnothing \in \tau$,
2. if $v \in \tau$ and $v = uj$ for some $j \in \mathbb{N}^*$, then $u \in \tau$,
3. for every $u \in \tau$, there exists an integer $k_u(\tau) \geq 0$ (the number of children of u) such that, for every $j \in \mathbb{N}^*$, $uj \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$.

In the following, by *tree* we will always mean plane tree. We denote the set of all trees by $\mathbf{T}$. We will often view each vertex of a tree τ as an individual of a population whose τ is the genealogical tree. If τ is a tree and $u \in \tau$, we define the shift of τ at u by $\sigma_u \tau = \{v \in U : uv \in \tau\}$, which is itself a tree. If $u, v \in \tau$ we denote by $[\![u, v]\!]$ the discrete geodesic path between u and v in τ. The total progeny of τ, which is the total number of vertices of τ, will be denoted by $\zeta(\tau)$. The number of leaves (vertices u of τ such that $k_u(\tau) = 0$) of the tree τ is denoted by $\lambda(\tau)$. Finally, we let $\mathbf{T}_n^{(\ell)}$ denote the set of all plane trees with n leaves such that there is no vertex with exactly one child.

The following proposition is an easy combinatorial property, whose proof is omitted.

Proposition 2.2. *The duality application ϕ is a bijection between $\mathbf{D}_n$ and $\mathbf{T}_n^{(\ell)}$.*

The dissection is conversely easily obtained from the tree as shown in Fig. 4.

Finally, we briefly recall the standard definition of Galton-Watson trees. Let ρ be a probability measure on $\mathbb{N}$ such that $\rho(1) < 1$. The law of the Galton-Watson tree with offspring distribution ρ is the unique probability measure $\mathbb{P}_\rho$ on $\mathbf{T}$ such that:

1. $\mathbb{P}_\rho(k_\varnothing = j) = \rho(j)$ for $j \geq 0$,
2. for every $j \geq 1$ with $\rho(j) > 0$, conditionally on $\{k_\varnothing = j\}$, the subtrees $\sigma_1\tau, \ldots, \sigma_j\tau$ are i.i.d. with distribution $\mathbb{P}_\rho$.

It is well-known that if ρ has mean less than or equal to 1 then a ρ-Galton-Watson tree is almost surely finite. In the sequel, for every integer $j \geq 1$, $\mathbb{P}_{\rho,j}$ will stand for the probability measure on $\mathbf{T}^j$ which is the distribution of j independent trees of law $\mathbb{P}_\rho$. The canonical element of $\mathbf{T}^j$ will be denoted by $\mathfrak{f}$. For $\mathfrak{f} = (\tau_1, \ldots, \tau_j) \in \mathbf{T}^j$, let $\lambda(\mathfrak{f}) = \lambda(\tau_1) + \cdots + \lambda(\tau_j)$ be the total number of leaves of $\mathfrak{f}$.

2.2. Uniform Dissections Are Conditioned Galton-Watson Trees

In the rest of this work, $\mathcal{D}_n$ is a random dissection uniformly distributed over $\mathbf{D}_n$. We also set $\mathcal{T}n = \phi(\mathcal{D}_n)$ to simplify notation. Remark that $\mathcal{T}n$ is a random tree which belongs to $\mathbf{T}_n^{(\ell)}$.

Fix $c \in (0, 1/2)$ and define a probability distribution $\mu^{(c)}$ on $\mathbb{N}$ as follows:

$$\mu_0^{(c)} = \frac{1-2c}{1-c}, \quad \mu_1^{(c)} = 0, \quad \mu_i^{(c)} = c^{i-1} \text{ for } i \geq 2.$$

It is straightforward to check that $\mu^{(c)}$ is a probability measure, which moreover has mean equal to 1 when $c = 1 - 2^{-1/2}$. In the latter case, we drop the exponent (c) in the notation, so that $\mu := \mu^{(1-1/\sqrt{2})}$. The following theorem gives a connection between uniform dissections of P_n and Galton-Watson trees conditioned on their number of leaves. This connection has been obtained independently of the present work in [21].

Proposition 2.3. *The conditional probability distribution $\mathbb{P}_{\mu^{(c)}}(\,\cdot \mid \lambda(\tau) = n)$ does not depend on the choice of $c \in (0, 1/2)$ and coincides with the distribution of the dual tree $\mathcal{T}n$ of a uniformly distributed dissection of P_{n+1}.*

Proof. We adapt the proof of [15, Proposition 1.8] in our context. By Proposition 2.2, it is sufficient to show that for every $c \in (0, 1/2)$ the probability distribution $\mathbb{P}_{\mu^{(c)}}(\cdot \mid \lambda(\tau) = n)$ is the uniform probability distribution over $\mathbf{T}_n^{(\ell)}$. If τ is a tree, we denote by $u_0, \ldots, u_{\zeta(\tau)-1}$ the vertices of τ listed in lexicographical order and recall that k_{u_i} stands for the number of children of u_i. Let $\tau_0 \in \mathbf{T}_n^{(\ell)}$. By the definition of $\mathbb{P}_{\mu^{(c)}}$, we have

$$\mathbb{P}_{\mu^{(c)}}(\tau = \tau_0 \mid \lambda(\tau) = n) = \frac{1}{\mathbb{P}_{\mu^{(c)}}(\lambda(\tau) = n)} \prod_{i=0}^{\zeta(\tau_0)-1} \mu_{k_{u_i}}^{(c)}.$$

Using the definition of $\mu^{(c)}$, the product appearing in the last expression can be written as

$$\prod_{i=0}^{\zeta(\tau_0)-1} \mu_{k_{u_i}}^{(c)} = \left(\frac{1-2c}{1-c}\right)^{\lambda(\tau_0)} c^{\zeta(\tau_0)-1-(\zeta(\tau_0)-\lambda(\tau_0))} = c^{-1} \left(\frac{c(1-2c)}{1-c}\right)^{\lambda(\tau_0)}.$$

Thus $\mathbb{P}_{\mu^{(c)}}(\tau = \tau_0 \mid \lambda(\tau) = n)$ depends only on $\lambda(\tau_0)$. We conclude that $\mathbb{P}_{\mu^{(c)}}(\,\cdot \mid \lambda(\tau) = n)$ is the uniform distribution over $\mathbf{T}_n^{(\ell)}$. ∎

In the following, we will always choose $c = 1 - 2^{-1/2}$ for $\mu^{(c)} = \mu$ to be critical. Hence, the random tree $\mathcal{T}n$ has law $\mathbb{P}_\mu(\cdot \mid \lambda(\tau) = n)$. A general study of Galton-Watson trees conditioned by their number of leaves is made in [14]. In particular, we will make an extensive use of the following asymptotic estimate which is a particular case of [14, Theorem 3.1]:

Lemma 2.4. *We have*

$$\mathbb{P}_\mu\left(\lambda(\tau)=n\right) \underset{n\to\infty}{\sim} \frac{n^{-3/2}}{2\sqrt{\pi\sqrt{2}}}. \tag{2}$$

Let us give an application of Proposition 2.3 and Lemma 2.4 to the enumeration of dissections. There exists no easy closed formula for the number $\#\mathbf{D}_n$ of dissections of P_{n+1}. However, a recursive decomposition easily shows that the generating function

$$D(z) := \sum_{n\geq 3} z^n \#\mathbf{D}_{n-1},$$

is equal to $\frac{z}{4}(1+z-\sqrt{z^2-6z+1})$, see e.g. [4, Section 3] and [10]. Using classical techniques of analytic combinatorics [11], it is then possible to get the asymptotic behavior of $\#\mathbf{D}_n$, see [10]. Here, we present a very short "probabilistic" proof of this result.

Corollary 2.5 (Flajolet & Noy, [10])**.** *We have*

$$\#\boldsymbol{D}_{n-1} \underset{n\to\infty}{\sim} \frac{1}{4}\sqrt{\frac{99\sqrt{2}-140}{\pi}}\,n^{-3/2}(3+2\sqrt{2})^n.$$

Proof. Let $n\geq 3$ and let $\tau_0=\{\varnothing,1,2,\ldots,n-1\}$ be the tree consisting of the root and its $n-1$ children. By Proposition 2.3, we have

$$\frac{1}{\#\mathbf{D}_{n-1}} = \mathbb{P}_\mu(\tau=\tau_0\,|\,\lambda(\tau)=n-1) = \frac{\mathbb{P}_\mu(\tau=\tau_0)}{\mathbb{P}_\mu(\lambda(\tau)=n-1)} = \frac{\mu_{n-1}\mu_0^{n-1}}{\mathbb{P}_\mu(\lambda(\tau)=n-1)}.$$

Thus

$$\#\mathbf{D}_{n-1} = \frac{\mathbb{P}_\mu(\lambda(\tau)=n-1)}{(2-\sqrt{2})^{n-1}\left(\frac{2-\sqrt{2}}{2}\right)^{n-2}} = \frac{(2-\sqrt{2})^3}{4}\frac{\mathbb{P}_\mu(\lambda(\tau)=n-1)}{(3-2\sqrt{2})^n}. \tag{3}$$

The statement of the corollary now follows from (2) and (3). ∎

2.3. Non-Crossing Trees Are *Almost* Conditioned Galton-Watson Trees

Definition 2.6. A *non-crossing tree* $\mathscr{C}$ of P_n is a tree drawn in the plane whose vertices are all the vertices of P_n and whose edges are Euclidean line segments that do not intersect except possibly at their endpoints.

Every non-crossing tree $\mathscr{C}$ inherits a plane tree structure by rooting $\mathscr{C}$ at the vertex 1 of P_n and keeping the planar ordering induced on $\mathscr{C}$. The children of the root vertex are ordered by going in clockwise order around the point 1 of P_n, starting from the edge connecting 1 to $e^{-2i\pi/n}$, which may or may not be in $\mathscr{C}$. As in [20], we call this plane tree *the shape* of $\mathscr{C}$ and denote it by $S(\mathscr{C})$. Obviously $\zeta(S(\mathscr{C}))=n$. Note that the mapping $\mathscr{C}\mapsto S(\mathscr{C})$ is not one-to-one. However, we will later see that large scale properties of uniform non-crossing trees are governed by their shapes.

In the following, we let $\mathscr{C}_n$ be uniformly distributed over the set of all non-crossing trees of P_n. We also set $\mathscr{T}_n = S(\mathscr{C}_n)$ to simplify notation. We start by recalling a result of Marckert and Panholzer stating that $\mathscr{T}_n$ is *almost* a Galton-Watson tree. Consider the two offspring distributions:

$$\begin{aligned} \nu_\varnothing(k) &= 2 \cdot 3^{-k}, && \text{for } k = 1, 2, 3, \ldots \\ \nu(k) &= 4(k+1)3^{-k-2}, && \text{for } k = 0, 1, 2, \ldots \end{aligned}$$

Following [20], we introduce a *modified* version of the ν-Galton-Watson tree where the root vertex has a number of children distributed according to $\nu_\varnothing$ and all other individuals have offspring distribution ν. We denote the resulting probability measure on plane trees obtained by $\widetilde{\mathbb{P}}_\nu$. The following theorem is the main result of [20] and will be useful for our purposes:

Theorem 2.7 (Marckert & Panholzer, [20]). *The random plane tree $\mathscr{T}_n$ is distributed according to $\widetilde{\mathbb{P}}_\nu(\cdot \mid \zeta(\tau) = n)$.*

3. THE BROWNIAN TRIANGULATION: A UNIVERSAL LIMIT FOR RANDOM NON-CROSSING CONFIGURATIONS

Recall that $\mathcal{D}_n$ is a uniform dissection of P_{n+1} and that $\mathcal{T}_n$ stands for its dual plane tree. Recall also that $\mathscr{C}_n$ is a uniform non-crossing tree of P_n and that $\mathscr{T}_n$ stands for its shape. In the following, we will view both $\mathcal{D}_n$ and $\mathscr{C}_n$ as random closed subsets of $\overline{\mathbb{D}}$ as suggested by Fig. 1. Recall that the *Hausdorff distance* between two closed subsets of $A, B \subset \overline{\mathbb{D}}$ is

$$\mathrm{d_{Haus}}(A,B) = \inf\left\{\varepsilon > 0 : A \subset B^{(\varepsilon)} \text{ and } B \subset A^{(\varepsilon)}\right\},$$

where $X^{(\varepsilon)}$ is the ε-enlargement of a set $X \subset \overline{\mathbb{D}}$. The set of all closed subsets of $\overline{\mathbb{D}}$ endowed with the Hausdorff distance is a compact metric space. Recall that the Brownian triangulation $\mathcal{B}$ is defined by (1). The main result of this section is:

Theorem 3.1. *The following two convergences in distribution hold for the Hausdorff metric on closed subsets of $\overline{\mathbb{D}}$:*

$$(i) \quad \mathcal{D}_n \xrightarrow[n\to\infty]{(d)} \mathcal{B}, \qquad\qquad (ii) \quad \mathscr{C}_n \xrightarrow[n\to\infty]{(d)} \mathcal{B}.$$

The main ingredient in the proof of Theorem 3.1 is a scaling limit theorem for functions coding the trees $\mathcal{T}_n$ and $\mathscr{T}_n$. In order to state this result, let us introduce the contour function associated to a plane tree.

Fix a tree τ and consider a particle that starts from the root and visits continuously all the edges of τ at unit speed (assuming that every edge has unit length). When leaving a vertex, the particle moves towards the first non visited child of this vertex if there is such a child, or returns to the parent of this vertex. Since all the edges will be crossed twice, the total time needed to explore the tree is $2(\zeta(\tau) - 1)$. For $0 \le t \le 2(\zeta(\tau) - 1)$, $C_\tau(t)$ is defined as the distance to the root of the position of the particle at time t. For technical reasons, we set $C_\tau(t) = 0$ for $t \in [2(\zeta(\tau) - 1), 2\zeta(\tau)]$. The function $C_\tau(\cdot)$ is called the contour function of the tree τ. See [16] for a rigorous definition. For $t \in [0, 2(\zeta(\tau) - 1)]$ and $u \in \tau$, we say that the contour process visits the vertex u at time t if the particle is at u at time t. Similarly, if we say that the contour process visits an edge $\mathfrak{e}$ if the particle belongs to $\mathfrak{e}$ at time t.

Let $\mathbb{e}$ bet the normalized excursion of linear Brownian motion. The following convergences in distribution will be useful for our purposes:

$$\left(\frac{C_{\mathcal{T}_n}(2\zeta(\mathcal{T}_n)t)}{\sqrt{n}}\right)_{0\le t\le 1} \xrightarrow[n\to\infty]{(d)} \left((3\sqrt{2}-4)^{-1/2}\mathbb{e}(t)\right)_{0\le t\le 1}, \tag{4}$$

$$\left(\frac{C_{\mathscr{T}_n}(2nt)}{\sqrt{n}}\right)_{0\le t\le 1} \xrightarrow[n\to\infty]{(d)} \left(2\sqrt{\frac{2}{3}}\mathbb{e}(t)\right)_{0\le t\le 1}. \tag{5}$$

The convergence (4) has been proved by Kortchemski [14, Theorem 5.9, Remark 5.10], and (5) has been obtained by Marckert and Panholzer [20, Proposition 4]. Although not relevant for our purposes, let us mention that it follows from [14, Corollary 3.3] that $\zeta(\mathcal{T}_n)$ is concentrated around $\mu_0^{-1}n$ as $n \to \infty$ and thus (4) means that the global shape of $\mathcal{T}_n$ is the same as that of a μ-Galton-Watson tree conditioned on having $\lfloor \mu_0^{-1}n \rfloor$ vertices in total.

The proof of Theorem 3.1 will be different for dissections and non-crossing trees, although the main ideas are the same in both cases. Notice for example that in the case of non-crossing trees, there is no need to consider a dual structure since the shape of the non-crossing tree already furnishes a plane tree.

3.1. Large Uniform Dissections

3.1.1. The Brownian Triangulation is the Limit of Large Uniform Dissections. In [15], a general convergence result is proved for dissections whose dual tree is a conditioned Galton-Watson tree whose offspring distribution belongs to the domain of attraction of a stable law. Since our approach to the convergence of large uniform non-crossing trees towards the Brownian triangulation will be similar in spirit, we reproduce the main steps of the proof in our particular case.

The following lemma, which is an easy consequence of [14, Corollary 3.3], roughly says that leaves are distributed uniformly in a conditioned Galton-Watson tree. Formally, if τ is a plane tree, for $0 \le t \le 2\zeta(\tau) - 2$, we let $\Lambda_\tau(t)$ be the number of leaves among the vertices of τ visited by the contour process up to time t, and we set $\Lambda_\tau(t) = \lambda(\tau)$ for $2\zeta(\tau) - 2 \le t \le 2\zeta(\tau)$.

Lemma 3.2. *We have*

$$\sup_{0\le t\le 1}\left|\frac{\Lambda_{\mathcal{T}n}\ (2\zeta(\mathcal{T}n)t)}{n} - t\right| \xrightarrow[n\to\infty]{(\mathbb{P})} 0, \tag{6}$$

where $(\mathbb{P})$ *stands for the convergence in probability.*

Proof of Theorem 3.1 part (i). We can apply Skorokhod's representation theorem (see e.g. [5, Theorem 6.7]) and assume, without loss of generality, that the convergences (4) and (6) hold almost surely and we aim at showing that $\mathcal{D}_n$ converges almost surely towards the Brownian triangulation $\mathcal{B}$ defined by (1). Since the space of compact subsets of $\overline{\mathbb{D}}$ equipped with the Hausdorff metric is compact, it is sufficient to show that the sequence $(\mathcal{D}_n)_{n\ge 1}$ has a unique accumulation point which is $\mathcal{B}$. We fix ω such that both convergences (4) and (6) hold for this value of ω. Up to extraction, we thus suppose that $(\mathcal{D}_n)_{n\ge 1}$ converges towards a certain compact subset $\mathcal{D}_\infty$ of $\overline{\mathbb{D}}$ and we aim at showing that $\mathcal{D}_\infty = \mathcal{B}$.

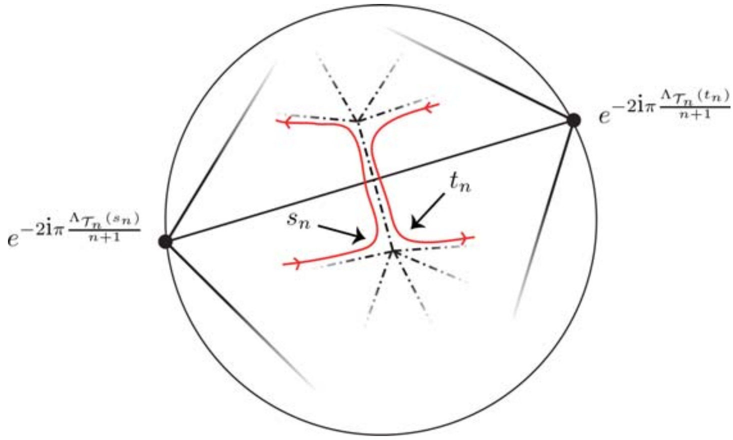

Fig. 5. The arrows show the first visit time s_n and last visit time t_n of an edge of $\mathcal{T}_n$. [Color figure can be viewed in the online issue, which available at wileyonlinelibrary.com.]

We first show that $\mathcal{B} \subset \mathcal{D}_\infty$. Fix $0 < s < t < 1$ such that $\mathbb{e}(s) = \mathbb{e}(t) = \min_{[s\wedge t, s\vee t]} \mathbb{e}$. We first consider the case when we have also $\mathbb{e}(r) > \mathbb{e}(s)$ for every $r \in (s, t)$. Let us prove that $[e^{-2i\pi s}, e^{-2i\pi t}] \subset \mathcal{D}_\infty$. Using the convergence (4), one can find an edge of $\mathcal{T}n$ such that if s_n is the time of the first visit of this edge by the contour process and if t_n is the time of its last visit, then $s_n/(2\zeta(\mathcal{T}_n)) \to s$ and $t_n/(2\zeta(\mathcal{T}_n)) \to t$ as $n \to \infty$, see Fig. 5.

Since the sides of P_{n+1}, excepting the side connecting 1 to $e^{2i\pi/(n+1)}$, are in one-to-one correspondence with the leaves of $\mathcal{T}_n$ we have (see Fig. 5):

$$\left[e^{-2i\pi\frac{\Lambda_{\mathcal{T}n}(s_n)}{n+1}}, e^{-2i\pi\frac{\Lambda_{\mathcal{T}n}(t_n)}{n+1}}\right] \in \mathcal{D}_n.$$

We refer to [15] for a more complete proof. From Lemma 3.2, we can pass to the limit and obtain $[e^{-2i\pi s}, e^{-2i\pi t}] \subset \mathcal{D}_\infty$.

Let us now suppose that $\mathbb{e}(s) = \mathbb{e}(t) = \min_{[s\wedge t, s\vee t]} \mathbb{e}$ and, moreover, there exists $r \in (s, t)$ such that $\mathbb{e}(r) = \mathbb{e}(s)$. Since local minima of Brownian motion are distinct, there exist two sequences of real numbers $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ taking values in $[0, 1]$ such that $\alpha_n \to s$, $\beta_n \to t$ as $n \to \infty$ and such that for every $n \geq 1$ and $r \in (\alpha_n, \beta_n)$ we have $\mathbb{e}(r) > \mathbb{e}(\alpha_n) = \mathbb{e}(\beta_n)$. The preceding argument yields $[e^{-2i\pi\alpha_n}, e^{-2i\pi\beta_n}] \subset \mathcal{D}_\infty$ for every $n \geq 1$. Since $\mathcal{D}_\infty$ is closed, we conclude that $\mathcal{B} \subset \mathcal{D}_\infty$.

The reverse inclusion is obtained by making use of a maximality argument. More precisely, it is easy to show that $\mathcal{D}_\infty$ is a *lamination*, that is a closed subset of $\overline{\mathbb{D}}$ which can be written as a union of chords that do not intersect each other inside $\mathbb{D}$. However, the Brownian triangulation $\mathcal{B}$, which is also a lamination, is almost surely maximal for the inclusion relation among the set of all laminations of $\overline{\mathbb{D}}$, see [17]. It follows that $\mathcal{D}_\infty = \mathcal{B}$. This completes the proof of the theorem. ∎

3.1.2. Application to the Study of the Number of Intersections with a Given Chord. We now explain how the ingredients of the previous proof can be used to study the number of intersections of a large dissection with a given chord. For $\alpha, \beta \in [0, 1]$, we denote by $I_n^{\alpha,\beta}$ the number of intersections of $\mathcal{D}_n$ with the chord $[e^{-2i\pi\alpha}, e^{-2i\pi\beta}]$, with the convention $I_n^{\alpha,\beta} = 0$ if $[e^{-2i\pi\alpha}, e^{-2i\pi\beta}] \subset \mathcal{D}_n$.

Proposition 3.3. *For $0 < \alpha < \beta < 1$ we have*

$$\frac{I_n^{\alpha,\beta}}{\sqrt{n}} \xrightarrow[n\to\infty]{(d)} \frac{\mathbb{e}(\beta-\alpha)}{\sqrt{3\sqrt{2}-4}},$$

where $\mathbb{e}$ is the normalized excursion of linear Brownian motion.

Proof. For $1 \le i \le n$, denote by $l^n(i)$ the i-th leaf of $\mathcal{T}n$ in the lexicographical order. Then, for $1 \le i < j \le n$, the construction of the dual tree shows that for every $s \in \left(\frac{i-1}{n+1}, \frac{i}{n+1}\right)$ and $t \in \left(\frac{j-1}{n+1}, \frac{j}{n+1}\right)$, $I_n^{s,t}$ is equal to the graph distance in the tree $\mathcal{T}n$ between the leaves $l^n(i)$ and $l^n(j)$. Indeed, the edges of $\mathcal{D}_n$ that intersect the chord $[e^{-2i\pi\alpha}, e^{-2i\pi\beta}]$ correspond exactly to the edges composing the shortest path between $l^n(i)$ and $l^n(j)$ in $\mathcal{T}_n$. However, the situation is more complicated when $e^{-2i\pi s}$ or $e^{-2i\pi t}$ coincides with a vertex of P_{n+1}. To avoid these particular cases, we note that for $\varepsilon, \varepsilon' \in (-\frac{1}{n+1}, \frac{1}{n+1})$ we have

$$\left| I_n^{s+\varepsilon,t+\varepsilon'} - I_n^{s,t} \right| \le 2\Delta^{(n)},$$

where $\Delta^{(n)}$ is the maximal number of diagonals of $\mathcal{D}_n$ adjacent to a vertex of P_{n+1}. We claim that

$$\frac{\Delta^{(n)}}{\sqrt{n}} \xrightarrow[n\to\infty]{(\mathbb{P})} 0. \tag{7}$$

We leave the proof of the claim to the reader since a (much) stronger result will be given in Theorem 4.7. Let $0 < \alpha < \beta < 1$. Set $i_n = \lfloor (n+1)\alpha \rfloor + 1$ and $j_n = \lfloor (n+1)\beta \rfloor + 1$. Choose n sufficiently large so that $j_n < n$. The preceding discussion shows that

$$\left| I_n^{\alpha,\beta} - \mathrm{d_{gr}}(l^n(i_n), l^n(j_n)) \right| \le 2\Delta^{(n)}, \tag{8}$$

where $\mathrm{d_{gr}}$ stands for the graph distance between two vertices in $\mathcal{T}n$. Now note that the graph metric of the tree $\mathcal{T}n$ can be recovered from the contour function of $\mathcal{T}n$, see [9]: If u_n, v_n are two vertices of $\mathcal{T}n$ such that the contour process reaches u_n (resp. v_n) at the instant s_n (resp. t_n), then

$$\mathrm{d_{gr}}(u_n, v_n) \quad = \quad C_{\mathcal{T}n}(s_n) + C_{\mathcal{T}n}(t_n) - 2 \inf_{u \in [s_n \wedge t_n, s_n \vee t_n]} C_{\mathcal{T}n}(u). \tag{9}$$

If we choose $u_n = l^n(i_n)$ and $v_n = l^n(j_n)$ with respective first visit times s_n and t_n, Lemma 3.2 shows that $s_n/2\zeta(\mathcal{T}n) \to \alpha$ and $t_n/2\zeta(\mathcal{T}n) \to \beta$ in probability as $n \to \infty$. Consequently, using (7), (8), together with (4), and (9) we finally obtain

$$\frac{I_n^{\alpha,\beta}}{\sqrt{n}} \quad \xrightarrow[n\to\infty]{(d)} \quad (3\sqrt{2}-4)^{-1/2} \left(\mathbb{e}(\alpha) + \mathbb{e}(\beta) - 2 \inf_{[\alpha\wedge\beta,\alpha\vee\beta]} \mathbb{e} \right).$$

To conclude, observe that by the re-rooting property of the Brownian excursion (see [19, Proposition 4.9]), the variable $\mathbb{e}(\alpha) + \mathbb{e}(\beta) - 2\inf_{[\alpha\wedge\beta,\alpha\vee\beta]} \mathbb{e}$ has the same distribution as $\mathbb{e}(\beta-\alpha)$. ■

Remark 3.4. The preceding proof can be adapted easily to show the following functional convergence in distribution

$$\left(\frac{I_n^{\alpha,\beta}}{\sqrt{n}} \right)_{\substack{0\le\alpha\le1 \\ 0\le\beta\le1}} \xrightarrow[n\to\infty]{(d)} (3\sqrt{2}-4)^{-1/2} \cdot \left(\mathbb{e}(\alpha) + \mathbb{e}(\beta) - 2 \inf_{[\alpha\wedge\beta,\alpha\vee\beta]} \mathbb{e} \right)_{\substack{0\le\alpha\le1 \\ 0\le\beta\le1}}.$$

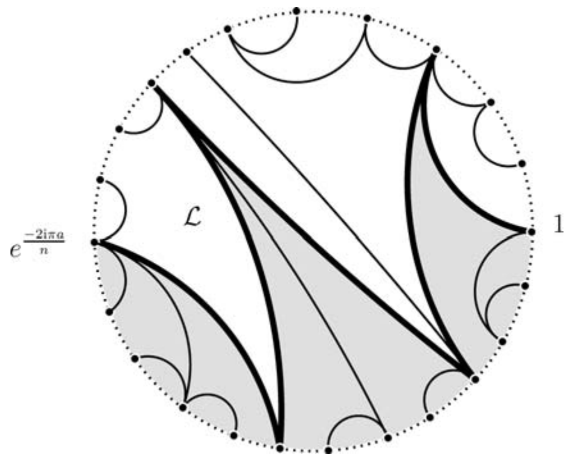

Fig. 6. Illustration of the proof of Lemma 3.5. We represent the non-crossing tree $\mathscr{C}$ with curved chords for better visibility. The left-hand side of $\mathcal{L}$ is in gray whereas its right-hand side is in white.

3.2. Large Uniform Non-Crossing Trees

In order to study large uniform non-crossing trees, the following lemma will be useful. It roughly implies that the location of a leaf in a non-crossing tree $\mathscr{C}$ can be deduced from its location in the shape of $\mathscr{C}$ up to an error that is bounded by its height. Recall that if τ is a tree and $u \in \tau$, k_u denotes the number of children of u.

Lemma 3.5. *Let $\mathscr{C}$ be a non-crossing tree with n vertices and shape $S(\mathscr{C}) = \tau$. Fix a vertex $u \in \tau$ and let $a \in \{0, 1, \ldots, n-1\}$ be such that the vertex in $\mathscr{C}$ corresponding to u is* $\exp(-2i\pi a/n)$. *Then there exists $i_0 \in \{1, \ldots, k_u + 1\}$ such that*

$$|a - \#\{v \in \tau : v \prec ui_0\}| \leq |u|,$$

where $\prec$ stands for the strict lexicographical order on $\mathcal{U}$.

Proof. Let $u \in \tau\backslash\{\varnothing\}$. Consider the discrete geodesic path from $\varnothing$ to u in τ and its image $\mathcal{L}$ in $\mathscr{C}$. There exists $1 \leq i_0 \leq k_u + 1$ such that, in $\mathscr{C}$, the first $i_0 - 1$ children of u as well as their descendants are folded on the left of $\mathcal{L}$ (oriented from the root) and the rest of the descendants of u are folded on the right of $\mathcal{L}$, see Fig. 6. Now, consider the set $E = \{1, \exp(-2i\pi/n), \ldots, \exp(-2i\pi a/n)\}$ of all the vertices of P_n that are between 1 and $\exp(-2i\pi a/n)$ in clockwise order. A geometric argument (see Fig. 6) shows that if a vertex x of $\mathscr{C}$ belongs to E, then its corresponding vertex in the tree τ must belong to the set $\{v \in \tau : v \prec ui_0\}$. On the other hand, if $w \in \{v \in \tau : v \prec ui_0\}$ and if, moreover, w is not a strict ancestor of u in τ then its corresponding vertex in $\mathscr{C}$ belongs to E. Consequently, we have

$$\#\{v \in \tau : v \prec ui_0\} - |u| + 1 \quad \leq \quad \#E = a + 1 \quad \leq \quad \#\{v \in \tau : v \prec ui_0\}.$$

The lemma follows (the case $u = \varnothing$ being trivial). ∎

For our purpose, it will be convenient to reinterpret this lemma using the contour function. Fix a tree τ, and define $\mathcal{Z}_\tau(j)$ as the number of distinct vertices of τ visited by the contour process of τ up to time j, for $0 \le j \le 2\zeta(\tau)-2$. For technical reasons, we set $\mathcal{Z}_\tau(j) = \zeta(\tau)$ for $j = 2\zeta(\tau) - 1$ and $j = 2\zeta(\tau)$, and then extend $\mathcal{Z}_\tau(\cdot)$ to the whole segment $[0, 2\zeta(\tau)]$ by linear interpolation. Note that a vertex $u \in \tau$ with k_u children is visited exactly $k_u + 1$ times by the contour function of τ, and that if $t^{(1)}, \dots, t^{(k_u+1)}$ are these times, then for every $i_0 \in \{1, \dots, k_u + 1\}$ we have

$$\#\{v \in \tau : v \prec ui_0\} = \mathcal{Z}_\tau\left(t^{(i_0)}\right). \tag{10}$$

The idea of the proof of Theorem 3.1 part (ii) is the following. Let $\mathscr{C}$ be a large non-crossing tree with shape $S(\mathscr{C})$. Pick a vertex $u \in S(\mathscr{C})$ corresponding to the point $\exp(-2ia/n)$ in $\mathscr{C}$. The goal is to recover a (with error at most $o(n)$) from the knowledge of $S(\mathscr{C})$ and u. Assume that u is a leaf of $S(\mathscr{C})$. Then u is visited only once by the contour process, say at time t_u. By Lemma 3.5 the quantity $|a - \mathcal{Z}_{S(\mathscr{C})}(t_u)|$ is less than the height of the tree $S(\mathscr{C})$ which is small in comparison with n by (5). Hence a in is known up to an error $o(n)$. We will see that the control by the leaves of $S(\mathscr{C})$ is sufficient for proving the convergence towards the Brownian triangulation.

The next lemma is an analogous to Lemma 3.2.

Lemma 3.6. *We have*

$$\sup_{0\le t\le 1}\left|\frac{\mathcal{Z}_{\mathcal{T}_n}(2nt)}{n} - t\right| \xrightarrow[n\to\infty]{(\mathbb{P})} 0. \tag{11}$$

Proof. This is a consequence of (5). Indeed, by [16, Eq. (13) in Section 1.6] (see also [8]):

$$\sup_{0\le t\le 1}\left|\frac{\mathcal{Z}_{\mathcal{T}_n}(2nt)}{n} - t\right| \le \frac{1}{2n}\sup_{0\le t\le 1} C_{\mathcal{T}_n}(2nt) + \frac{1}{n} = \frac{1}{2\sqrt{n}}\sup_{0\le t\le 1}\frac{C_{\mathcal{T}_n}(2nt)}{\sqrt{n}} + \frac{1}{n} \quad \xrightarrow[n\to\infty]{(\mathbb{P})} \quad 0.$$

∎

Proof of Theorem 3.1 part (ii). Similarly to the proof of part (i) of the theorem, we can apply Skorokhod's theorem and assume that the convergences (5) and (11) hold almost surely. We fix ω such that both convergences (5) and (11) hold for this value of ω. Up to extraction, we thus suppose that $(\mathscr{C}_n)_{n\ge 1}$ converges towards a compact subset $\mathscr{C}_\infty$ of $\overline{\mathbb{D}}$ and we aim at showing that $\mathscr{C}_\infty = \mathcal{B}$.

We first show that $\mathcal{B} \subset \mathscr{C}_\infty$. Fix $0 < s < t < 1$ such that $\mathbb{e}(s) = \mathbb{e}(t) = \min_{[s\wedge t, s\vee t]} \mathbb{e}$ and assume furthermore that $\mathbb{e}(r) > \mathbb{e}(s)$ for $r \in (s,t)$. Let us show that $[e^{-2i\pi s}, e^{-2i\pi t}] \subset \mathscr{C}_\infty$. To this end, we fix $\varepsilon > 0$ and show that $[e^{-2i\pi s}, e^{-2i\pi t}] \subset \mathscr{C}_n^{(6\varepsilon)}$ for n sufficiently large (recall that $X^{(\varepsilon)}$ is the ε-enlargement of a closed subset $X \subset \overline{\mathbb{D}}$). Using the convergence (5), for every n large enough, one can find integers $0 \le s_n < t_n \le 2n - 2$ such that if u_n (resp. v_n) denotes the vertex of $\mathcal{T}_n$ visited at time s_n (resp. t_n) by the contour process, the following three properties are satisfied:

- $s_n/2n \to s$, $t_n/2n \to t$,
- u_n and v_n are leaves in $\mathcal{T}_n$,
- for every vertex w_n in $[\![u_n, v_n]\!]$ (the discrete geodesic path between u_n and v_n in $\mathcal{T}_n$) and every visit time r_n of w_n by the contour process, we have

$$\min\left(\left|\frac{r_n}{2n} - s\right|, \left|\frac{r_n}{2n} - t\right|\right) \le \varepsilon. \tag{12}$$

For the second property, we can for instance use the fact that local maxima of $\mathbb{e}$ are dense in $[0, 1]$. We now claim that under these assumptions, for n large enough, the image $\mathcal{L}_n$ in $\mathscr{C}_n$ of the discrete geodesic path $[\![u_n, v_n]\!]$ in $\mathscr{T}_n$ lies within Hausdorff distance 6ε from the line segment $[e^{-2i\pi s}, e^{-2i\pi t}]$.

Indeed, let $w_n \in [\![u_n, v_n]\!]$ and let $a_n \in \{0, 1, \ldots, n-1\}$ such that the vertex of $\mathscr{C}_n$ corresponding to w_n is $z_n = \exp(-2i\pi a_n/n)$. Applying Lemma 3.5 to $u = w_n$ and using (10) we can find a time r_n at which the contour process is at w_n and such that

$$|a_n - \mathcal{Z}_{\mathscr{T}_n}(r_n)| \leq |w_n|.$$

By the convergence (11) and the bound (12), there exists an integer $N \geq 1$, independent of the choice of w_n, such that for $n \geq N$

$$\min\left(\left|\frac{\mathcal{Z}_{\mathscr{T}_n}(r_n)}{n} - s\right|, \left|\frac{\mathcal{Z}_{\mathscr{T}_n}(r_n)}{n} - t\right|\right) \leq 2\varepsilon.$$

On the other hand, thanks to the convergence (5) we have

$$\frac{1}{n} \sup_{u \in \mathscr{T}_n} |u| = \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} \sup_{0 \leq t \leq 1} C_{\mathscr{T}_n}(2nt) \quad \xrightarrow[n\to\infty]{} \quad 0.$$

We conclude that there exists an integer $N' \geq 1$, independent of the choice of w_n, such that for $n \geq N'$, we have $\min(|\frac{a_n}{n} - s|, |\frac{a_n}{n} - t|) \leq 3\varepsilon$ and thus that $\min\left(|z_n - e^{-2i\pi s}|, |z_n - e^{-2i\pi t}|\right) \leq 6\varepsilon$. It follows that for large n, we have

$$\mathcal{L}_n \subset [e^{-2i\pi s}, e^{-2i\pi t}]^{(6\varepsilon)}. \tag{13}$$

On the other hand, u_n and v_n are leaves, so they are visited at a unique time by the contour process. By the same arguments, for every n sufficiently large, we deduce that their images α_n and β_n in $\mathscr{C}_n$ satisfy $|\alpha_n - e^{-2i\pi s}| \leq 6\varepsilon$ and $|\beta_n - e^{-2i\pi t}| \leq 6\varepsilon$. Consequently, since $\mathcal{L}_n$ is a finite union of line segments connecting α_n to β_n, we deduce from (13) that for every n sufficiently large enough

$$[e^{-2i\pi s}, e^{-2i\pi t}] \subset \mathcal{L}_n^{(6\varepsilon)} \subset \mathscr{C}_n^{(6\varepsilon)}. \tag{14}$$

The case when there exists $r \in (s, t)$ such that $\mathbb{e}(s) = \mathbb{e}(r) = \mathbb{e}(t)$ (this r is then a.s. unique by standard properties of the Brownian excursion) is treated exactly as in the proof of the first assertion of this theorem. We conclude that $\mathcal{B} \subset \mathscr{C}_\infty$. The reverse inclusion is obtained by making use of a maximality argument, see part (i). ∎

3.3. Universality of the Brownian Triangulation and Applications

The convergence in distribution of random compact subsets towards the Brownian triangulation yields information on their asymptotic geometrical properties that are preserved under the Hausdorff convergence. Let us give an example of application of this fact.

Let χ_n be a random configuration on the vertices of P_n, that is a random closed subset made of line segments connecting some of the vertices of the polygon. Assume that χ_n converges in distribution towards $\mathcal{B}$ in the sense of the Hausdorff metric. Let $\mathrm{diag}(\chi_n)$ be the angle of the longest diagonal of χ_n (by definition, the angle of $[e^{-2i\pi s}, e^{-2i\pi t}]$ with

$0 \leq s \leq t \leq 1$ is $\min(t-s, 1-t+s)$). Then, as $n \to \infty$, the law of $\text{diag}(\chi_n)$ converges in distribution towards the angle of the longest diagonal of χ_n, given by:

$$\frac{1}{\pi}\frac{3x-1}{x^2(1-x)^2\sqrt{1-2x}}\mathbf{1}_{\frac{1}{3}\leq x\leq\frac{1}{2}}\,\mathrm{d}x.$$

This distribution has been computed in [3] (see also [7]). The preceding convergence follows from the fact that the length of the longest chord is a continuous function of configurations for the Hausdorff metric. Similar limit theorems hold for a large variety of other functionals, such as the area of the face with largest area, etc.

It is plausible that many other uniformly distributed non-crossing configurations (see [10]) converge towards the Brownian triangulation in the Hausdorff sense. We give here a few instances of this phenomenon.

3.3.1. Dissections With Constrained Face Degrees

Theorem 3.7. *Let $\mathcal{A}$ be a non-empty subset of $\{3,4,5,\ldots\}$. Let $\mathbf{D}_n^{(\mathcal{A})}$ be the set of all dissections of P_{n+1} whose face degrees all belong to the set $\mathcal{A}$. We restrict our attention to the values of n for which $\mathbf{D}_n^{(\mathcal{A})} \neq \varnothing$.*

1. *There exists a probability distribution $\nu_{\mathcal{A}}$ on $\mathbb{N}$ such that if $\sigma_{\mathcal{A}}^2$ denotes the variance of $\nu_{\mathcal{A}}$, we have*

$$\#\mathbf{D}_{n-1}^{(\mathcal{A})} \underset{n\to\infty}{\sim} \sqrt{\frac{\nu_{\mathcal{A}}(2)^4\nu_{\mathcal{A}}(0)^3}{2\pi\sigma_{\mathcal{A}}^2}} \cdot \frac{n^{-3/2}}{(\nu_{\mathcal{A}}(2)\nu_{\mathcal{A}}(0))^n}.$$

2. *Let $\mathcal{D}_n^{(\mathcal{A})}$ be uniformly distributed over $\mathbf{D}_n^{(\mathcal{A})}$. Then $\mathcal{D}_n^{(\mathcal{A})}$ converges towards the Brownian triangulation.*

Note that the case $\mathcal{A} = \{3\}$ corresponds to uniform triangulations and the case $\mathcal{A} = \{3,4,5,\ldots\}$ corresponds to uniform dissections.

Proof. The proof of this statement goes along the very same lines as the proofs of Corollary 2.5 and Theorem 3.1 (i) by noticing that the dual tree $\phi(\mathcal{D}_n^{(\mathcal{A})})$ is a Galton-Watson tree conditioned on having n leaves for a certain finite variance offspring distribution $\nu_{\mathcal{A}}$. More precisely, if we denote the set $\{a-1 : a \in \mathcal{A}\}$ by $\mathcal{A}-1$, let $c_{\mathcal{A}} \in (0,1)$ be the unique real number in $(0,1)$ such that

$$\sum_{i\in\mathcal{A}-1} i c_{\mathcal{A}}^{i-1} = 1.$$

Then $\nu_{\mathcal{A}}$ is defined by

$$\nu_{\mathcal{A}}(0) = 1 - \sum_{i\in\mathcal{A}-1} c_{\mathcal{A}}^{i-1}, \quad \nu_{\mathcal{A}}(i) = c_{\mathcal{A}}^{i-1} \text{ for } i \in \mathcal{A}-1.$$

Note that $\nu_{\mathcal{A}}(2) = c_{\mathcal{A}}$ and that $\nu_{\mathcal{A}}$ automatically has a finite variance $\sigma_{\mathcal{A}}^2 > 0$. ∎

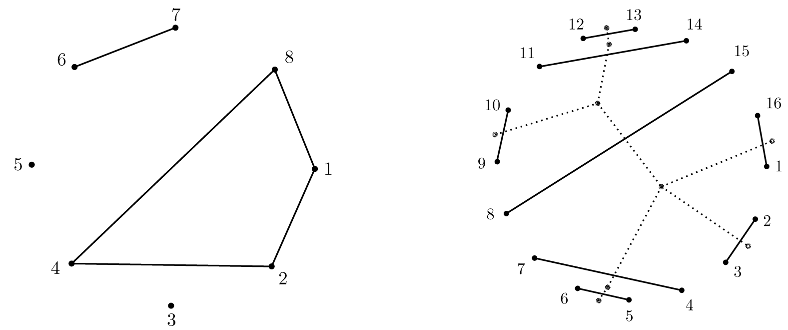

Fig. 7. A non-crossing partition of P_8 and a non-crossing pair-partition of P_{16} together with its dual tree.

3.3.2. Non-Crossing Graphs. A *non-crossing graph* of P_n is a graph drawn on the plane, whose vertices are the vertices of P_n and whose edges are non-crossing line segments. Let $\mathcal{G}_n$ be uniformly distributed over the set of all non-crossing graphs of P_n. Note that $\mathcal{G}_n$ can be seen as a compact subset of $\overline{\mathbb{D}}$. Then $\mathcal{G}_n$ converges in distribution towards the Brownian triangulation.

This fact easily follows from the convergence of uniform dissections towards the Brownian triangulation. Indeed, if $\mathcal{G}$ is a non-crossing graph of P_n, let $\psi(\mathcal{G})$ be the compact subset of $\overline{\mathbb{D}}$ obtained from $\mathcal{G}$ by adding the sides of P_n. As noticed at the end of Section 3.1 in [10], $\psi(\mathcal{G})$ is a dissection, and every dissection has 2^n pre-images by ψ. It follows that the random dissection $\psi(\mathcal{G}_n)$ is a uniform dissection of P_n. The conclusion follows, since the Hausdorff distance between $\psi(\mathcal{G}_n)$ and $\mathcal{G}_n$ tends to 0.

3.3.3. Non-Crossing Partitions and Non-Crossing Pair Partitions. A non-crossing partition of P_n is a partition of the vertices of P_n (labeled by the set $\{1, 2, \ldots, n\}$) such that the convex hulls of its blocks are pairwise disjoint (see Fig. 7 where the partition $\{\{1, 2, 4, 8\}, \{3\}, \{5\}, \{6, 7\}\}$ is represented). A non-crossing pair-partition of P_n is a non-crossing partition of P_n whose blocks are all of size 2 (see Fig. 7 where the pair-partition $\{\{1, 16\}, \{2, 3\}, \{4, 7\}, \{5, 6\}, \{8, 15\}, \{9, 10\}, \{11, 14\}, \{12, 13\}\}$ is represented).

Let $\mathcal{P}_n^{(2)}$ be a uniformly distributed random variable on the set of all non-crossing pair-partitions of P_{2n}, seen as a compact subset of $\overline{\mathbb{D}}$. Then $\mathcal{P}_n^{(2)}$ converges towards the Brownian triangulation.

To establish this fact, we rely once again on a coding of $\mathcal{P}_n^{(2)}$ by a critical Galton-Watson tree. One easily sees that the dual tree of $\mathcal{P}_n^{(2)}$ (see Fig. 7) is a uniform tree with n edges, which is also well-known to be a Galton-Watson tree with geometric offspring distribution, conditioned on having n edges. One can then show the convergence of $\mathcal{P}_n$ towards the Brownian triangulation using the same methods as in the case of uniform dissections. Details are left to the reader.

Let us now discuss non-crossing partitions. Let $\mathcal{P}_n$ be a uniformly distributed random variable on the set of all non-crossing partitions of P_n, and view $\mathcal{P}_n$ as a random compact subset of $\overline{\mathbb{D}}$. Then $\mathcal{P}_n$ converges towards the Brownian triangulation.

This follows from the convergence of non-crossing pair-partitions. Indeed, given a non-crossing pair-partition of P_{2n}, we get a non-crossing partition of P_n by identifying the n pairs of vertices of the form $(2i-1, 2i)$ for $1 \leq i \leq n$ (see Fig. 7 where the non-crossing partition

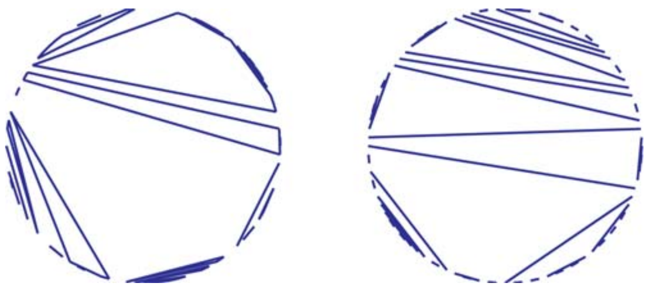

Fig. 8. Uniform non-crossing partition and pair-partition of P_{100}. [Color figure can be viewed in the online issue, which available at wileyonlinelibrary.com.]

of P_8 is obtained by contraction of vertices from the non-crossing pair-partition of P_{16}). This identification gives a bijection between non-crossing partitions of P_n and non-crossing pair partitions of P_{2n} and the desired result easily follows (Fig. 8).

Let us state the previously discussed results in a global statement:

Theorem 3.8. *The following convergences hold in distribution for the Hausdorff metric on closed subsets of* $\overline{\mathbb{D}}$:

$$\mathcal{G}_n \xrightarrow[n\to\infty]{(d)} \mathcal{B}, \qquad \mathcal{P}_n^{(2)} \xrightarrow[n\to\infty]{(d)} \mathcal{B}, \qquad \mathcal{P}_n \xrightarrow[n\to\infty]{(d)} \mathcal{B}.$$

At first sight, it may seem mysterious that Galton-Watson trees appear behind many different models of uniform non-crossing configurations. In [10], using suitable parameterizations, Flajolet and Noy manage to find a Lagrange inversion-type implicit equation for the generating functions of these configurations. Generating functions verifying a Lagrange inversion-type implicit equation are those of simply-generated trees, which are very closely related to Galton-Watson trees (see [1, Section 2.1]). This explains why Galton-Watson trees are hidden behind various models of uniformly distributed non-crossing configurations.

4. GRAPH-THEORETICAL PROPERTIES OF LARGE UNIFORM DISSECTIONS

In this section, we study graph-theoretical properties of large dissections using the Galton-Watson tree structure identified in Proposition 2.3. Let us stress that as in Proposition 3.7, all the results contained in this section can be adapted easily to uniform dissections constrained on having all face degrees in a fixed non-empty subset of $\{3, 4, \ldots\}$ (and in particular to uniform triangulations).

As previously, $\mathcal{D}_n$ is a uniformly distributed dissection of P_{n+1} and $\mathcal{T}_n$ denotes its dual tree with n leaves. We start by recalling the definition and the construction of the so-called critical Galton-Watson tree conditioned to survive.

4.1. The Critical Galton-Watson Tree Conditioned to Survive

If τ is a tree and k is a nonnegative integer, we let $[\tau]_k = \{u \in \tau : |u| \leq k\}$ denote the tree obtained from τ by keeping the vertices in the first k generations. Let $\xi = (\xi_i)_{i\geq 0}$ be an offspring distribution with $\xi_1 \neq 1$ and $\sum i\xi_i = 1$. We denote by T_n a Galton-Watson tree

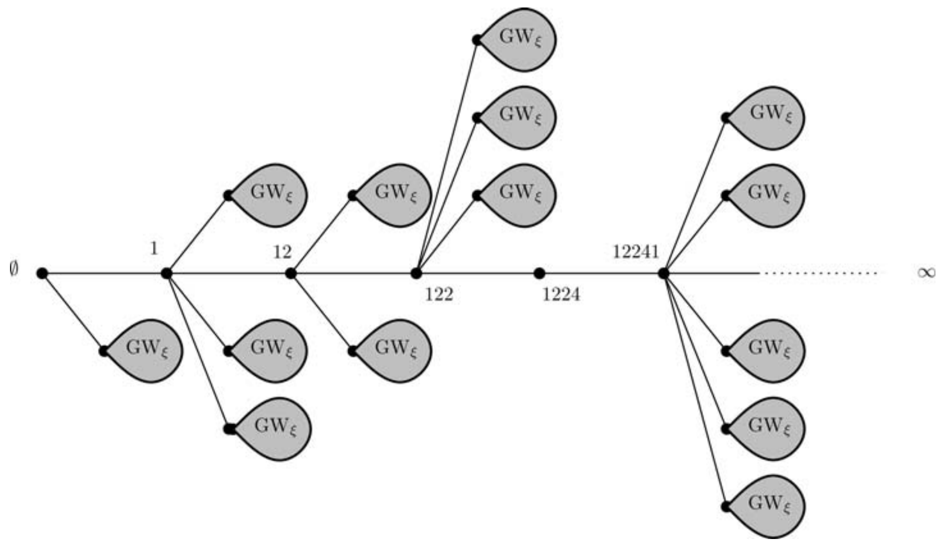

Fig. 9. An illustration of T_∞ under $\widehat{\mathbb{P}}_\xi$.

with offspring distribution ξ conditioned on having height at least $n \geq 0$. Kesten [13, Lemma 1.14] showed that for every $k \geq 0$, we have the following convergence in distribution

$$[T_n]_k \xrightarrow[n\to\infty]{(d)} [T_\infty]_k\,,$$

where T_∞ is a random infinite plane tree called the critical ξ-Galton-Watson tree conditioned to survive.

We denote the law of the ξ-Galton-Watson tree conditioned to survive by $\widehat{\mathbb{P}}_\xi$ and by T_∞ a random tree distributed according to $\widehat{\mathbb{P}}_\xi$. Let us describe this tree (see [13, 18]). We let $\overline{\xi}$ be the size-biased distribution of ξ defined by $\overline{\xi}_k = k\xi_k$ for $k \geq 0$. Let $(D_i)_{i\geq 0}$ be a sequence of i.i.d. random variables distributed according to $\overline{\xi}$. Let also $(U_i)_{i\geq 1}$ be a sequence of random variables such that, conditionally on $(D_i)_{i\geq 0}$, $(U_i)_{i\geq 1}$ are independent and U_{k+1} is uniformly distributed over $\{1, 2, \ldots, D_k\}$ for every $k \geq 0$. The tree T_∞ has a unique spine, that is a unique infinite path $(\varnothing, U_1, U_1U_2, U_1U_2U_3, \ldots) \in \mathbb{N}^{*\mathbb{N}^*}$ and the degree of $U_1U_2 \ldots U_k$ is D_k. Finally, conditionally on $(U_i)_{i\geq 1}$ and $(D_i)_{i\geq 0}$ all the remaining subtrees are independent ξ-Galton-Watson trees, see Fig. 9.

The critical Galton-Watson tree conditioned to survive also arises in other conditionings of Galton-Watson trees. Recall that $\mathcal{T}n$ is a μ-Galton-Watson tree conditioned on having n leaves.

Theorem 4.1. *For every $k \geq 0$, we have the convergence in distribution*

$$[\mathcal{T}n]_k \xrightarrow[n\to\infty]{(d)} [\mathcal{T}_\infty]_k\,,$$

where $\mathcal{T}_\infty$ is the critical Galton-Watson tree with offspring distribution μ conditioned to survive.

Remark 4.2. Theorem 4.1 is true when μ is replaced by any finite variance offspring distribution ν such that $\mathbb{P}_\nu\,(\lambda(\tau) = n) > 0$ for every n large enough.

Proof. This follows from another description of the law $\widehat{\mathbb{P}}_\mu$: If τ_0 is a plane tree and $k \geq 0$ is an integer, we denote by $L_k(\tau_0)$ the number of individuals of τ_0 at height exactly k. Then, if τ_0 has height k, we have from [13, Lemma 1.14]:

$$\widehat{\mathbb{P}}_\mu([\tau]_k = \tau_0) = L_k(\tau_0)\mathbb{P}_\mu([\tau]_k = \tau_0).$$

Fix an integer $k \geq 1$ as well as a tree $\tau_0 \in \mathbf{T}$ of height k. In order to prove the theorem, it is thus sufficient to show that

$$\mathbb{P}_\mu([\tau]_k = \tau_0 \,|\, \lambda(\tau) = n) \underset{n\to\infty}{\longrightarrow} L_k(\tau_0)\mathbb{P}_\mu([\tau]_k = \tau_0).$$

Denote by q the number of leaves of τ_0 that have a height strictly smaller than k. By the branching property of Galton-Watson trees, we have

$$\mathbb{P}_\mu([\tau]_k = \tau_0 \,|\, \lambda(\tau) = n) = \mathbb{P}_\mu([\tau]_k = \tau_0)\frac{\mathbb{P}_{\mu,L_k(\tau_0)}(\lambda(\mathfrak{f}) = n-q)}{\mathbb{P}_\mu(\lambda(\tau) = n)},$$

where we recall that $\mathbb{P}_{\mu,i}$ denotes the probability distribution of a forest of i independent Galton-Watson trees with law $\mathbb{P}_\mu$. Since q and $L_k(\tau_0)$ are fixed, it is thus sufficient to show that for a fixed integer $i \geq 1$, as $n \to \infty$

$$\mathbb{P}_{\mu,i}(\lambda(\mathfrak{f}) = n) \underset{n\to\infty}{\sim} i \times \mathbb{P}_\mu(\lambda(\tau) = n).$$

This follows from the next lemma, which we state in a more general form than needed here in view of further applications. ∎

Lemma 4.3. *There exists $\varepsilon > 0$ such that if $(i_n)_{n\geq 1}$ is a sequence of positive integers such that $i_n \leq n^\varepsilon$ for every $n \geq 1$, we have*

$$\mathbb{P}_{\mu,i_n}(\lambda(\mathfrak{f}) = n) \underset{n\to\infty}{\sim} i_n \cdot \mathbb{P}_\mu(\lambda(\tau) = n).$$

Proof. We show that the conclusion of the lemma holds for any $\varepsilon \in (0, 1/9)$. To simplify notation, we set $p_k := \mathbb{P}_\mu(\lambda(\tau) = k)$ for every integer $k \geq 1$ and write $i = i_n$ in the proof. By the definition of $\mathbb{P}_{\mu,i}$, we have

$$\mathbb{P}_{\mu,i}(\lambda(\mathfrak{f}) = n) = \sum_{k_1+\ldots+k_i=n} \prod_{j=1}^{i} p_{k_j}.$$

We will show that when n is large, the main contribution in the previous sum is obtained when the indices $k_1, \ldots, k_i$ are such that only one of them is of order n and the others are small in comparison. Let $A \geq 1$. Firstly, notice that at least one of the indices $k_1, \ldots, k_i$ is larger than n/i. Secondly, let us evaluate the contribution of the sum when $k_1 \geq n/i$ and k_2 is larger than A. By Lemma 2.4,

$$\sum_{\substack{k_1+\cdots+k_i=n \\ k_1\geq n/i \\ k_2 \geq A}} \prod_{j=1}^{i} p_{k_j} \leq \sup_{k_1\geq n/i} p_{k_1} \cdot \left(\sum_{k_2\geq A} p_{k_2}\right) \cdot \prod_{j=3}^{i}\left(\sum_{k_j=1}^{\infty} p_{k_j}\right) \leq C(n/i)^{-3/2}A^{-1/2},$$

for some constant $C > 0$ which is independent of n and i. Hence, provided that $A < n^{1-\varepsilon}$,

$$\left| \left(\sum_{k_1+\cdots+k_i=n} \prod_{j=1}^{i} p_{k_j} \right) - i \left(\sum_{1\le k_1,\ldots,k_{i-1}\le A} p_{n-\sum_j k_j} \prod_{j=1}^{i-1} p_{k_j} \right) \right| \le C i^{7/2} n^{-3/2} A^{-1/2}. \tag{15}$$

We apply this to $A = A(n) = n^{8\varepsilon}$ in such a way that the right-hand side of the above inequality is negligible in comparison with ip_n. Then note that for $1 \le k_1, \ldots, k_{i-1} \le A$, we have

$$n - n^{9\varepsilon} \le n - \sum_{j=1}^{i-1} k_j \le n.$$

Moreover, since $9\varepsilon < 1$, Lemma 2.4 gives $p_j/p_n \to 1$ uniformly in $n - n^{9\varepsilon} \le j \le n$ as $n \to \infty$. Thus, using Lemma 2.4 again

$$i \left(\sum_{1\le k_1,\ldots,k_{i-1}\le A} p_{n-\sum_j k_j} \prod_{j=1}^{i-1} p_{k_j} \right) \sim ip_n \left(\sum_{k=1}^{n^{8\varepsilon}} p_k \right)^{i-1} \sim ip_n \left(1 - \frac{n^{-4\varepsilon}}{\sqrt{\pi\sqrt{2}}} \right)^{i} \sim ip_n,$$

which completes the proof of the lemma. ■

4.2. Applications

The "local convergence" given in Theorem 4.1 allows us to study "local" properties of large uniform dissections by reading them directly on the critical Galton-Watson tree conditioned to survive. We will focus our attention on the following two local properties of random uniform dissections: Vertex degrees and face degrees.

Let us introduce some notation. Recall that $\mathcal{D}_n$ stands for a uniformly distributed dissection of P_{n+1}. Denote by $\delta^{(n)}$ the degree of the face adjacent to the side $[1, e^{2i\pi/(n+1)}]$ in the random dissection $\mathcal{D}_n$ and by $D^{(n)}$ the maximal degree of a face of $\mathcal{D}_n$. Similarly, denote by $\partial^{(n)}$ the number of diagonals adjacent to the vertex corresponding to the complex number 1 in $\mathcal{D}_n$ and by $\Delta^{(n)}$ the maximal number of diagonals adjacent to some vertex of P_{n+1}. Finally, for $b > 0$, we write $\log_b(\cdot)$ for $\ln(\cdot)/\ln(b)$.

We shall establish that $\delta^{(n)}$ and $\partial^{(n)}$ converge in distribution, and read their limiting distributions on the random infinite tree $\mathcal{T}_\infty$. We also provide sharp concentration estimates on $D^{(n)}$ and $\Delta^{(n)}$, confirming in particular a conjecture of [4] concerning $\Delta^{(n)}$.

4.2.1. Face Degrees

Proposition 4.4. *As n goes to infinity, $\delta^{(n)}$ converges in distribution to the random variable X with distribution*

$$\mathbb{P}(X = k) = (k-1)\mu_{k-1} = (k-1)\left(\frac{2-\sqrt{2}}{2}\right)^{k-2}, \qquad k \ge 3.$$

Proof. This is an immediate consequence of Theorem 4.1 and the construction of the critical Galton-Watson tree conditioned to survive, after taking into account the fact that $\delta^{(n)} - 1$ is the number of children of $\varnothing$ in the dual tree of $\mathcal{D}_n$. ■

Proposition 4.5. *Set* $\beta = 1/\mu_2 = 2 + \sqrt{2}$. *For every* $c > 0$, *we have*

$$\mathbb{P}\left(\log_\beta(n) - c\log_\beta\log_\beta(n) \le D^{(n)} \le \log_\beta(n) + c\log_\beta\log_\beta(n)\right) \xrightarrow[n\to\infty]{} 1.$$

Proof. By construction of the dual tree $\mathcal{T}_n$ of $\mathcal{D}_n$, we have $D^{(n)} - 1 = \max_{u\in\mathcal{T}n} k_u$. Thus, by Proposition 2.3, for every measurable function $F : \mathbb{R}_+ \to \mathbb{R}_+$ we have

$$\mathbb{E}\left[F(D^{(n)} - 1)\right] = \mathbb{E}_\mu\left[F\left(\max_{u\in\tau} k_u\right) \mid \lambda(\tau) = n\right].$$

The result now follows from [14, Remark 7.3]. ∎

4.2.2. Vertex Degrees. We are now interested in another graph-theoretical property of large uniform dissections, namely vertex degrees. Since these vertex degrees are read on the dual tree in a more complicated fashion than face degrees, the arguments are slightly more involved.

Recall that $\partial^{(n)}$ stands for the number of diagonals adjacent to the vertex corresponding to the complex number 1 in the uniform dissection $\mathcal{D}_n$.

Proposition 4.6. *As n goes to infinity,* $\partial^{(n)}$ *converges in distribution to the sum of two independent geometric random variables of parameter* $1 - \mu_0 = \sqrt{2} - 1$, *i.e. for any* $k \ge 0$ *we have*

$$\mathbb{P}(\partial^{(n)} = k) \xrightarrow[n\to\infty]{} (k+1)\mu_0^2(1-\mu_0)^k.$$

See also [4] for a closely related result.

Proof. If τ is a plane tree, we introduce the length $\ell(\tau)$ of the left-most path in τ starting at $\varnothing$ (that is following left-most children until we reach a leaf),

$$\ell(\tau) = \max\{i \ge 0 : 1_i \in \tau\}, \quad \text{where } 1_i = 1\dots1 \quad (i \text{ times}) \text{ with } 1_0 = \varnothing.$$

Using the bijection between a dissection and its dual tree, it is easy to see that the number of diagonals adjacent to the vertex 1 of the random dissection $\mathcal{D}_n$ is $\ell(\mathcal{T}_n) - 1$. By Theorem 4.1, for every $k \ge 1$, $[\mathcal{T}n]_k \to [\mathcal{T}_\infty]_k$ as $n \to \infty$. It follows that $\partial^{(n)} = \ell(\mathcal{T}_n) - 1$ converges in distribution towards $\ell(\mathcal{T}_\infty) - 1$. Let us identify the distribution of this variable using the first description of the law $\widehat{\mathbb{P}}_\mu$: The length $\ell(\mathcal{T}_\infty) - 1$ can be decomposed into

$$\ell(\mathcal{T}_\infty) - 1 = (\ell_1 - 1) + \ell_2,$$

where ℓ_1 is the smallest integer $i \ge 1$ such that the element $1_i = 1\dots1$ (i times) is not on the *spine* of $\mathcal{T}_\infty$ and $\ell_2 = \ell(\mathcal{T}_\infty) - (\ell_1 - 1)$ is the length of the left-most path in the critical μ-Galton-Watson tree grafted on the left of 1_i. By the description in Section 4.1, the two variables $\ell_1 - 1$ and ℓ_2 are independent. It is straightforward that ℓ_2 is distributed according to a geometric variable of parameter $\sqrt{2}-1$. Let us now turn to $\ell_1 - 1$. Recall the notation introduced in Section 4.1. For $k \geq 0$, we have:

$$\begin{aligned}\mathbb{P}(\ell_1 \geq k+1) &= \mathbb{P}(U_1 = 1, U_2 = 1, \ldots, U_k = 1)\\ &= \prod_{i=0}^{k-1}\left(\sum_{j=1}^{\infty}\mathbb{P}(D_i = j)\mathbb{P}(U_{i+1} = 1 \mid D_i = j)\right)\\ &= \left(\sum_{j=2}^{\infty}(1-2^{-1/2})^{j-1}\right)^k = (1-\mu_0)^k.\end{aligned}$$

We thus see that $\ell_1 - 1$ is also geometric with parameter $\sqrt{2}-1$ and the desired result follows. ∎

Recall that $\Delta^{(n)}$ stands for the maximal number of diagonals adjacent to a vertex of P_{n+1}. The following theorem proves a conjecture of [4].

Theorem 4.7. *Set $b = 1/(1-\mu_0) = \sqrt{2}+1$. For every $c > 0$, we have*

$$\mathbb{P}\left(\Delta^{(n)} \geq \log_b(n) + (1+c)\log_b\log_b(n)\right) \xrightarrow[n\to\infty]{} 0.$$

Proof. Let $p, n \geq 1$ be integers. By rotational invariance, the degrees of the vertices of $\mathcal{D}_n$ are identically distributed random variables. It follows that

$$\mathbb{P}\left(\Delta^{(n)} \geq p\right) \leq (n+1)\mathbb{P}\left(\partial^{(n)} \geq p\right).$$

We have already noticed that the number of diagonals adjacent to the vertex 1 of the random dissection $\mathcal{D}_n$ corresponds to $\ell(\mathcal{T}_n) - 1$, where $\ell(\mathcal{T}n)$ denotes the length of the left-most path in $\mathcal{T}_n$ starting at $\varnothing$. Thus, by Proposition 2.3,

$$(n+1)\mathbb{P}\left(\partial_0^{(n)} \geq p\right) = (n+1)\mathbb{P}_\mu\left(\ell(\tau) \geq p+1 \mid \lambda(\tau) = n\right)$$

We now estimate the right-hand side and show that it tends to 0 when $n \to \infty$ and $p = p_n = \log_b(n) + (1+c)\log_b\log_b(n)$ for $c > 0$. If $\ell(\tau) = p$, define $\theta(\tau) = k_\varnothing + k_1 + k_{1_2} + \ldots + k_{1_{p-1}} - p$ ($\theta(\tau)$ can be interpreted as the total number of those children of vertices in the left-most path that are not in that path). Note that under $\mathbb{P}_\mu$, $\ell(\tau)$ is distributed according to a geometric random variable of parameter $\sqrt{2}-1$. In particular, for $\alpha = |4/\log(1-\mu_0)|$,

$$n^3\mathbb{P}_\mu\left(\ell(\tau) \geq \alpha\log(n)\right) \quad \xrightarrow[n\to\infty]{} \quad 0. \tag{16}$$

Note also that for positive integers j, k we have $\mathbb{P}_\mu\left(\theta(\tau) = j \mid \ell(\tau) = k\right) = \mathbb{P}(Y^{*k} = j)$, where Y^{*k} is distributed as the sum of n independent random variables distributed according to $\gamma(j) = \mu_{j+1}/\mu([1,\infty])$.

Choose $\varepsilon > 0$ such that the conclusion of Lemma 4.3 holds. We first claim that if $A_n = \{\theta(\tau) \le n^\varepsilon\}$, then $\mathbb{P}_\mu(A_n) \ge 1 - n^{-3}$ for n large enough. To this end, write

$$\begin{aligned} n^3\mathbb{P}_\mu(\theta(\tau) > n^\varepsilon) &\le n^3\mathbb{P}_\mu(\ell(\tau) > \alpha\log(n)) \\ &\quad + n^3 \sum_{j=1}^{\lfloor \alpha \log(n) \rfloor} \mathbb{P}_\mu(\theta(\tau) > n^\varepsilon \,\|\, \ell(\tau) = j)\, \mathbb{P}_\mu(\ell(\tau) = j) \\ &\le n^3\mathbb{P}_\mu(\ell(\tau) > \alpha\log(n)) \\ &\quad + n^3\alpha\log(n)\mathbb{P}_\mu(\theta(\tau) > n^\varepsilon \,\|\, \ell(\tau) = \lfloor \alpha\log(n)\rfloor). \end{aligned}$$

The first term tends to zero by (16), and the second one as well by the previous description of the law of $\theta(\tau)$ under the conditional probability distribution $\mathbb{P}_\mu(\cdot \mid \ell(\tau) = k)$ and a standard large deviation inequality. Thus our claim holds and it follows that, for n large enough,

$$(n+1)\mathbb{P}_\mu(A_n^c \mid \lambda(\tau) = n) \le (n+1)n^{-3}/\mathbb{P}_\mu(\lambda(\tau) = n) \xrightarrow[n\to\infty]{} 0 \tag{17}$$

by Lemma 2.4. We now consider the event A_n. We have

$$\begin{aligned} &\mathbb{P}_\mu(\{\ell(\tau) = p\} \cap A_n \mid \lambda(\tau) = n) \\ &\quad \le \mu_0 \sum_{\substack{r_0,\ldots,r_{p-1}\ge 0 \\ \sum r_j \le n^\varepsilon}} \mu_{r_0+1}\cdots\mu_{r_{p-1}+1} \frac{\mathbb{P}_{\mu, r_0+r_1+\cdots+r_{p-1}}(\lambda(\tau) = n-1)}{\mathbb{P}_\mu(\lambda(\tau) = n)}. \end{aligned}$$

We can then apply Lemma 4.3 to get that the quotient in the last display is bounded above by some constant C_2 times $\theta(\tau) = r_0 + \ldots + r_p$, so that

$$\begin{aligned} \mathbb{P}_\mu(\ell(\tau) = p, \theta(\tau) \le n^\varepsilon \mid \lambda(\tau) = n) &\le C_2\mu_0 \sum_{r_0,\ldots,r_{p-1}\ge 0} \mu_{r_0+1}\cdots\mu_{r_{p-1}+1}(r_0 + \ldots + r_{p-1}) \\ &= C_2 p \mu_0^2 (1-\mu_0)^{p-1}, \end{aligned}$$

where we have successively calculated these sums by using

$$\sum_{k=1}^{\infty} \mu_{k+1} = 1 - \mu_0, \qquad \sum_{k=1}^{\infty} k\mu_{k+1} = \mu_0.$$

Consequently, setting $p_n = \log_b(n) + (1+c)\log_b\log_b(n)$ we deduce from the above estimates that

$$(n+1)\mathbb{P}_\mu(\ell(\tau) \ge p_n, \theta(\tau) \le n^\varepsilon \mid \lambda(\tau) = n) \xrightarrow[n\to\infty]{} 0,$$

which together with (17) completes the proof of the theorem. ▪

4.2.3. On $\log\log$ *Concentration.* In [4], it is shown that for every $c > 0$

$$\mathbb{P}\left(\Delta^{(n)} \le \log_b(n) - (2+c)\log_b\log_b(n)\right) \xrightarrow[n\to\infty]{} 0.$$

Using the connection between uniform dissections and Galton-Watson trees conditioned on their number of leaves, it is possible to refine the above lower bound and to replace $(2+c)$ by c. However, we believe that the optimal concentration result is given by the following conjecture:

Conjecture 4.8. *For every $c > 0$ we have*

$$\mathbb{P}\left(\left|\Delta^{(n)} - (\log_b(n) + \log_b \log_b(n))\right| > c \log_b \log_b(n)\right) \xrightarrow[n\to\infty]{} 0.$$

If the degrees of vertices in $\mathcal{D}_n$ were independent, this concentration result would hold. However, the difficulty comes from the fact that this independence property does not exactly hold. Let us mention that this conjecture (for a different value of b) has been proved in the particular case of triangulations using generating function methods [12].

It is worth pointing out that although δ^n and $\partial^{(n)}$ have a similar limiting distribution (roughly speaking a size-biased geometric distribution), the maximal degree of a face and the maximal degree of a vertex in a random uniform dissection possess a different concentration behavior: $D^{(n)}$ is strongly concentrated around $\log_\beta(n) + o(\log\log(n))$, whereas $\Delta^{(n)}$ is (conjecturally) strongly concentrated around $\log_b(n) + \log_b \log_b(n) + o(\log\log(n))$. This comes from the heuristic observation that a "typical" vertex has a limiting distribution which is given by a sized-biased geometric distribution, whereas a "typical" face of $\mathcal{D}_n$ has a limiting distribution which is a geometric distribution (which is not size-biased). Let us give some details.

We start with the vertex degree. By Proposition 4.6 and by rotational invariance, a "typical" vertex has a limiting distribution which is given by a sized-biased geometric distribution. This is why $\Delta^{(n)}$ should have the same concentration behavior as n independent random variables distributed as size-biased geometric laws, that is, $\Delta^{(n)}$ should be concentrated around $\log_b(n) + \log_b \log_b(n) + o(\log\log(n))$.

The situation is however different is the case of face degrees. Choosing the face adjacent to $[1, e^{2i\pi/(n+1)}]$ introduces a size-biasing in the distribution of a "typical" face (indeed, the face containing a given side of P_n is not a typical face but a size-biased one; a typical face would be a face chosen "uniformly" among all faces of the dissection). In other words, a typical face of $\mathcal{D}_n$ follows a geometric distribution, but $\delta^{(n)}$ is a size-biased distribution of a typical face of $\mathcal{D}_n$. This is why $D^{(n)}$ follows the same concentration behavior as n independent geometric variables (which are not size-biased), and hence explains why $D^{(n)}$ is concentrated around $\log_\beta(n) + o(\log\log n)$.

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