Good sequences, bijections and permutations

Igor Kortchemski[1]

Lycée Blaise Pascal, 91401 Orsay, France

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Abstract

In the present paper we study general properties of good sequences by means of a powerful and beautiful tool of combinatorics—the method of bijective proofs. A good sequence is a sequence of positive integers $k = 1, 2, \ldots$ such that the element k occurs before the last occurrence of $k + 1$. We construct two bijections between the set of good sequences of length n and the set of permutations of length n. This allows us to count good sequences as well as to calculate generating functions of statistics on good sequences. We study avoiding patterns on good sequences and discuss their relation with Eulerian polynomials. Finally, we describe particular interesting properties of permutations, again using bijections.

1 Introduction

Good sequences were introduced by Federico Ardila in the course of his statistical study of permutations [1]. He found the number of good sequences of a fixed length and proposed this problem as a question for the 2002 International Mathematical Olympiad. Later on, Richard P. Stanley reformulated this result as an exercise for the Clay Research Academy 2005 [2].

In this work we establish intimate relations between good sequences and permutations. To achieve this goal we construct a natural bijection—a standard method of combinatorics—between these two objects. Furthermore we study general properties of good sequences by calculating generating functions of certain statistics on good sequences. By the term statistic we mean a map between the set of good sequences and the set of positive integers. The simplest examples of statistics include the greatest element of a sequence and the sum of all the elements of the sequence. To perform the aforementioned calculation, we invoke a bijective argument to associate a statistic on good sequences to a statistic on permutations. The latter has already been calculated in references [2, 3, 4].

We demonstrate that the relative order of the elements of a good sequence is very similar to the order of the elements of a permutation. We use this fact to study avoiding patterns on good sequences. In the past, the pattern avoidance has been intensively studied in connection with permutations. Many interesting results have recently been discovered, among them the upper bound on the number of permutations avoiding a certain pattern [5, 6]. Several beautiful bijections have been constructed in this field, e.g, a correspondence has been established between

[1]E-mail address: `igork@free.fr`

123-avoiding permutations and Dyck paths [4]. This correspondence turns out to be useful in the study of avoiding patterns on good sequences. It indirectly leads us to a relation between Eulerian polynomials and the number of ways n competitors can rank in a competition, allowing for the possibility of ties. In addition, we present several combinatorial interpretations of Eulerian polynomials which play an important role in mathematics [7].

Last but not least, we employ good sequences to build a bijection from a set of permutations to itself. We study its action on ascents, descents, inversions, orbits and present its geometrical interpretation on permutations as rotations of the corresponding permutation matrix.

2 Good sequences and Bijections

2.1 Definition

Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called a *good sequence* if it satisfies the following condition: for each positive integer $k \geq 2$, if the number k appears in the sequence then so does the number $k-1$, and moreover the first occurrence of $k-1$ comes before the last occurrence of k.

For example :

2123 is a good sequence of length 4.
31312 is not a good sequence: 2 does not occur before the last occurrence of 3.

Let G_n be the set of all *good sequences* of fixed length n.

A few questions arise : *Is G_n finite? If yes, how many elements does it have?*
The following theorem will answer these questions.

2.2 Main Theorem

A few examples for low n:

n=1: $G_1 = \{1\}$
n=2: $G_2 = \{11, 12\}$
n=3: $G_3 = \{111, 112, 121, 122, 123, 212\}$
n=4: $G_4 = \{$1111, 1112,1121, 1122, 1123, 1211, 1212, 1213, 1221, 1222, 1223, 1231, 1232, 1233, 1234, 1323, 2112, 2121, 2122, 2123, 2132, 2212, 2312, 3123$\}$

so that

$$\mathrm{Card}(G_1) = 1\,, \quad \mathrm{Card}(G_2) = 2\,, \quad \mathrm{Card}(G_3) = 6\,, \quad \mathrm{Card}(G_4) = 24.$$

We notice that for these sets with low n, $\mathrm{Card}(G_n) = n!$. We shall prove that this relation holds for arbitrary n.

Theorem 2.1. *The number of good sequences of fixed length n is equal to $n!$.*

The proof consists in constructing a bijection $\mathcal{B}$ from G_n to the set of all permutations of $\{1, \ldots, n\}$. Since the number of elements of this set is $n!$, theorem 2.1 follows immediately.

The construction will be done in 3 steps.

2.2.1 Construction of $\mathcal{B}$

Definition: Let $[n]$ be the set $\{1,\ldots,n\}$ and let $\mathfrak{S}_n$ be the set of all *permutations* of $[n]$. Throughout the paper we will use the notation $a_1a_2\ldots a_k$ to denote a sequence of positive integers $a_1,a_2,\ldots,a_k$. We denote permutations as sequences, for instance, we will consider 564123 as a permutation of 123456.

Define $\mathcal{B} : G_n \to \mathfrak{S}_n$ by the following map: $\forall\ u \in G_n$, $\mathcal{B}(u) = a_1 \ldots a_n$ where a_i verifies:
a_1 is the position of the leftmost largest integer of u.
a_2 is the position of the second leftmost largest integer of u.
$\vdots$
a_i is the position of the i-th leftmost largest integer of u.
$\vdots$
a_n is the position of the n-th leftmost largest integer u.

Notice that by construction $a_1 \ldots a_n$ is a permutation of $[n]$.

<u>For example :</u>

$$\mathcal{B} : 2132 \mapsto 3142 \tag{1}$$
$$\mathcal{B} : 213421 \mapsto 431526 \tag{2}$$

2.2.2 Construction of $\mathcal{B}^{-1}$

To show that $\mathcal{B}$ is a bijection let us construct the inverse of $\mathcal{B}$.

Define $\mathcal{B}^{-1} : \mathfrak{S}_n \to G_n$ as the map: $\forall\ w = a_1 \ldots a_n \in \mathfrak{S}_n$, $\mathcal{B}^{-1}(w) = b_1 \ldots b_n$ where the b_i are defined recursively as follows:

$b_{a_n} = 1$
For i from n to 2 by -1

$$b_{a_{i-1}} = \begin{cases} b_{a_i} & \text{if } a_{i-1} < a_i \\ b_{a_i} + 1 & \text{if } a_{i-1} > a_i \end{cases}$$

Endfor

<u>For example :</u>

$\mathcal{B}^{-1} : \boxed{3142} \rightsquigarrow \cdot 1 \cdot \cdot \rightsquigarrow \cdot 1 \cdot 2 \rightsquigarrow 21 \cdot 2 \mapsto \boxed{2132}$, as in (1).
$\mathcal{B}^{-1} : \boxed{431526} \rightsquigarrow \cdot\cdot\cdot\cdot\cdot 1 \rightsquigarrow \cdot 1 \cdot\cdot\cdot 1 \rightsquigarrow \cdot 1 \cdot \cdot 21 \rightsquigarrow 21 \cdot \cdot 21 \rightsquigarrow 213 \cdot 21 \mapsto \boxed{213421}$, as in (2).

2.2.3 Proof that we obtain a bijection

We will first demonstrate the following proposition:

Proposition 2.2. *We have $\mathcal{B}^{-1} \circ \mathcal{B} = Id$.*

Lemma 2.3. *For $u \in G_n$ let m be the greatest element of u. Then for each $i \in [m]$, i is an element of u.*

Proof: It follows directly from the definition of a good sequence. ■

Let $u = b_1 \dots b_n$ and $\mathcal{B}(u) = a_1 \dots a_n$. For $i \in \{2, \dots, n\}$, we will show that if $a_{i-1} > a_i$ then $b_{a_{i-1}} = b_{a_i} + 1$ and if $a_{i-1} < a_i$ then $b_{a_{i-1}} = b_{a_i}$.
First, note that a_{i-1} appears before a_i in $\mathcal{B}(u)$. This means that $b_{a_{i-1}} \geq b_{a_i}$.

If $a_{i-1} > a_i$: $b_{a_{i-1}}$ and b_{a_i} cannot be equal, because a_i would have appeared before a_{i-1} in $\mathcal{B}(u)$, since its position is more to the left. Finally, if $a_{i-1} > a_i$, then $b_{a_{i-1}} > b_{a_i}$ and by lemma 2.3, $b_{a_{i-1}} = b_{a_i} + 1$.

If $a_{i-1} < a_i$: Suppose that $b_{a_{i-1}} > b_{a_i}$, that is $b_{a_{i-1}} = b_{a_i} + 1$ by lemma 2.3. Since $a_{i-1} < a_i$, b_{a_i} occurs only to the right of $b_{a_{i-1}}$.

On the other hand, $\forall j \in \{2, \dots, n\}$ such that $j \neq i$, $b_{a_{j-1}} \geq b_{a_j}$ and therefore we obtain:

$$b_{a_{i-1}} > b_{a_i} \geq b_{a_{i+1}} \dots \geq b_{a_n}.$$

Thus, a_{i-1} is the last position where $b_{a_{i-1}}$ occurs.

Therefore by virtue of previous remarks, b_{a_i} does not occur before the last occurrence of $b_{a_{i-1}} = b_{a_i} + 1$, which is a contradiction. Finally, if $a_{i-1} < a_i$, then $b_{a_{i-1}} = b_{a_i}$.

The smallest element of w is 1 by lemma 2.3, so that $b_{a_n} = 1$. By (2.2.2) we conclude that $\mathcal{B}^{-1} \circ \mathcal{B} = Id$. Hence $\mathcal{B}^{-1}$ is surjective and $\mathcal{B}$ is injective. ■

Proposition 2.4. *The map $\mathcal{B}^{-1}$ is injective.*

Proof: Let $w = a_1 \dots a_n \in \mathfrak{S}_n$. Suppose that w has exactly m descents. One can naturally split w into $m+1$ *subsequences* $w_1, \dots, w_{m+1}$ such that each w_i does not contain a descent, i.e., w_i is increasing and such that $w = w_1 \dots w_{m+1}$. Let $\mathcal{B}^{-1}(w) = b_1 \dots b_n$. By definition of the bijection $\mathcal{B}^{-1}$:

$$\begin{aligned}
b_i &= 1 \quad \text{for } i \in w_{m+1} \\
b_i &= 2 \quad \text{for } i \in w_m \\
&\vdots \\
b_i &= m+2-j \quad \text{for } i \in w_j \\
&\vdots \\
b_i &= m+1 \quad \text{for } i \in w_1.
\end{aligned} \tag{3}$$

Let w and w' be two permutations of length n. Hence $\mathcal{B}^{-1}(w) = \mathcal{B}^{-1}(w')$, if and only if w and w' have the same number of descents and exactly the same subsequences as defined above. In other words, $\mathcal{B}^{-1}(w) = \mathcal{B}^{-1}(w')$ if and only if $w = w'$. This proves that $\mathcal{B}^{-1}$ is injective. ■

We conclude that $\mathcal{B}^{-1}$ is a bijection. Hence $\mathcal{B}$ is a bijection too. Finally:

$$\boxed{\mathrm{Card}(G_n) = \mathrm{Card}(\mathfrak{S}_n) = n!}$$

■

3 Statistics on G_n

3.1 Descent and greatest element statistics

Definition: $w = a_1 \dots a_n \in \mathfrak{S}_n$. A *descent* of w is a number i for which $a_i > a_{i+1}$. E.g in $36\underline{7}2\underline{4}15$ there are 2 descents: 3 and 5.

Theorem 3.1. *The number of elements of $\mathfrak{S}_n$ with exactly m descents is equal to the number of elements of G_n with the greatest element $m+1$.*

Proof: To prove this theorem, let us make use of the bijection $\mathcal{B}$. Let $w = a_1 \dots a_n \in \mathfrak{S}_n$. Suppose that w has exactly m descents. One can naturally split w into $m+1$ *subsequences* $w_1, \dots, w_{m+1}$ such that each w_i does not contain a descent, i.e., w_i is increasing and such that $w = w_1 \dots w_{m+1}$. Let $\mathcal{B}^{-1}(w) = b_1 \dots b_n$. By definition of the bijection $\mathcal{B}^{-1}$:

$$\begin{aligned} b_i &= 1 \quad \text{for } i \in w_{m+1} \\ b_i &= 2 \quad \text{for } i \in w_m \\ &\vdots \\ b_i &= m+2-j \quad \text{for } i \in w_j \\ &\vdots \\ b_i &= m+1 \quad \text{for } i \in w_1. \end{aligned} \tag{4}$$

For example, for $w = 3672415$, one has $w_1 = 367$, $w_2 = 24$, $w_3 = 15$. One finds $\mathcal{B}^{-1}(3672415) := b_1 \dots b_7 = 1232133$ and indeed :

$$\begin{aligned} b_i &= 1 \quad \text{for } i \in \{1,5\} \\ b_i &= 2 \quad \text{for } i \in \{2,4\} \\ b_i &= 3 \quad \text{for } i \in \{3,6,7\}. \end{aligned}$$

Hence the greatest element of $\mathcal{B}^{-1}(w)$ is $m+1$. Since $\mathcal{B}$ is a bijection, the number of elements in G_n having this property is equal to the number of elements in $\mathfrak{S}_n$ having exactly m descents. ■

Definitions:

- Denote by $\mathrm{d}(w)$ the number of descents of w. $A_n(q)$ is called an *Eulerian polynomial* [3]. The coefficient of q^i in $A_n(q)$ is denoted $A(n,i)$ and satisfies the following relation:

$$\sum_{k \geq 0} k^n q^k = \frac{\sum_{i=1}^{n} A(n,i) q^i}{(1-q)^{n+1}}$$

  For instance:

$$\sum_{k \geq 0} k^4 q^k = \frac{q + 11q^2 + 11q^3 + q^4}{(1-q)^5}.$$

It can be proved (see [2]) that $A(n, k+1)$ counts the number of permutations of $[n]$ with k descents, that is:

$$A_n(q) = \sum_{w \in \mathfrak{S}_n} q^{1+\mathrm{d}(w)}.$$

For instance, the polynomial $A_4(q) = q + 11q^2 + 11q^3 + q^4$ for the set $\mathfrak{S}_4$ exhibits that the latter possesses one element with no descents, eleven with one descent, eleven with two descents and one with three descents.

- $u \in G_n$. Define $\max(u)$ as the greatest element of u.

Theorem 3.2. *The number of elements of G_n with the greatest element k is equal to $A(n, k)$ so that:*

$$\sum_{u \in G_n} q^{max(u)} = A_n(q).$$

Proof: By theorem 3.1, the number of elements of G_n with the greatest element k is equal to the number of elements of $\mathfrak{S}_n$ with $k-1$ descents, which is $A(n, k)$ by definition of the Eulerian polynomial. ■

Note that theorem 3.2 gives an alternative combinatorial view of the Eulerian polynomials.

3.2 Rank statistics

Definition: Let $u \in G_n$. Let k be the greatest element of u. For all $i \in [k]$ denote by n_i the number of occurrences of i in u. Define the *rank* of u by:

$$\mathrm{rank}(u) = \sum_{i=1}^{k} (i-1) n_i.$$

For instance, for $u = 423545122$ one finds $n_1 = 1$, $n_2 = 3$, $n_3 = 1$, $n_4 = 2$, $n_5 = 2$. So $\mathrm{rank}(423545122) = 0 \times 1 + 1 \times 3 + 2 \times 1 + 3 \times 2 + 4 \times 2 = 19$.

Theorem 3.3. *The following relation holds:*

$$\sum_{u \in G_n} q^{rank(u)} = (1+q)(1+q+q^2) \cdots (1+q+\ldots+q^{n-1}).$$

We will once again give a bijection proof, but before we turn to it, let us introduce one more definition:

Definition: $w \in \mathfrak{S}_n$. $\mathrm{des}(w)$ is the *descent set* of w, that is the set of all descents of w. E.g $\mathrm{des}(4\underline{6}1\underline{5}\underline{3}28\underline{9}7) = \{2, 4, 5, 8\}$. Define the *major index* of w by the sum of the elements of $\mathrm{des}(w)$:

$$\mathrm{maj}(w) = \sum_{i \in \mathrm{des}(w)} i.$$

For instance, $\mathrm{maj}(461352897) = 2 + 4 + 5 + 8 = 19$.

Lemma 3.4. *We have:* $\sum_{w\in\mathfrak{S}_n} q^{maj(w)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$.

A proof of this lemma can be found in [3].

Lemma 3.5. *Let* $w \in \mathfrak{S}_n$*. Then* $rank(\mathcal{B}^{-1}(w)) = maj(w)$*.*

For example, $\mathcal{B}^{-1}(461352897) = 423545122$ and that $\mathrm{rank}(423545122) = \mathrm{maj}(461352897) = 19$.

Proof of lemma 3.5: As in section 3.1, assume that w has exactly $k-1$ descents. Split w into k *subsequences* $w_1, \ldots, w_k$ such that each w_i does not contain a descent, i.e., w_i is increasing. Let $\mathcal{B}^{-1}(w) = b_1 \ldots b_n$, then by definition:

$$\begin{array}{rcl} b_i & = & 1 \quad \text{for } i \in w_k \\ b_i & = & 2 \quad \text{for } i \in w_{k-1} \\ & \vdots & \\ b_i & = & k+1-j \quad \text{for } i \in w_j \\ & \vdots & \\ b_i & = & k \quad \text{for } i \in w_1. \end{array}$$

Thus:

$$\begin{aligned} \mathrm{maj}(w) &= \mathrm{Card}(w_1) + \mathrm{Card}(w_1w_2) + \cdots + \mathrm{Card}(w_1w_2\ldots w_{k-1}) \\ &= \mathrm{Card}(w_1) + [\mathrm{Card}(w_1) + \mathrm{Card}(w_2)] + \cdots + [\mathrm{Card}(w_1) + \mathrm{Card}(w_2) + \cdots + \mathrm{Card}(w_{k-1})] \\ &= (k-1)\mathrm{Card}(w_1) + (k-2)\mathrm{Card}(w_2) + \cdots + 1 \times \mathrm{Card}(w_{k-1}) + 0 \times \mathrm{Card}(w_k) \\ &= (k-1)n_k + (k-2)n_{k-1} + \cdots + (2-1)n_2 + 0n_1 \\ &= \mathrm{rank}(\mathcal{B}^{-1}(w)). \end{aligned}$$

This proves lemma 3.5. ■

Proof of theorem 3.3: We obtain:

$$\sum_{u\in G_n} q^{\mathrm{rank}(u)} = \sum_{u\in G_n} q^{\mathrm{maj}(\mathcal{B}(u))} = \sum_{w\in\mathfrak{S}_n} q^{\mathrm{maj}(w)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

The first equality follows from lemma 3.5, the second from the fact that $\mathcal{B}$ is a bijection and the third one from lemma 3.4. ■

3.3 Degree statistics

Definition: Let $u \in G_n$. Define the *degree* of u as the sum of all the elements of u: $\deg(u) = \sum_{i\in u} i$.

The degree of u is closely related to the rank of u by the following argument: if k is the greatest element of u and if n_i is the number of occurrences of i in u then:

$$\deg(u) = \sum_{i=1}^{k} i\, n_i = \sum_{i=1}^{k} n_i(i-1) + \sum_{i=1}^{k} n_i = \mathrm{rank}(u) + n.$$

For instance, for $u = 423545122$ one finds $n_1 = 1$, $n_2 = 3$, $n_3 = 1$, $n_4 = 2$, $n_5 = 2$ so that $\mathrm{rank}(423545122) = 1 \times 1 + 3 \times 2 + 1 \times 3 + 2 \times 4 + 2 \times 5 = 28 = 19 + 9$.

Theorem 3.6. *We have:*

$$\sum_{u \in G_n} q^{\deg(u)} = q^n (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

Proof: By theorem 3.3:

$$\sum_{u \in G_n} q^{\deg(u)} = \sum_{u \in G_n} q^{\mathrm{rank}(u)+n} = q^n \sum_{u \in G_n} q^{\mathrm{rank}(u)} = q^n (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

3.4 Records

Definition: $u = a_1 \dots a_n$ is a sequence. A *record* of u is a term a_j such that $a_i < a_j$ for all $i < j$. Define $\mathrm{rec}(w)$ to be the total number of records of w and $\mathrm{srec}(w)$ to be the sum of positions of all records of w. E.g, in 47516823 there are 3 records which are 4, 7 and 8. Thus $\mathrm{rec}(47516823) = 3$ and $\mathrm{srec}(47516823) = 1 + 2 + 6 = 9$.

Theorem 3.7. *We have:*

$$\sum_{u \in G_n} q^{\mathrm{rec}(u)} = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{rec}(w)} = q(q+1)(q+2)\cdots(q+n-1)$$

and

$$\sum_{u \in G_n} q^{\mathrm{srec}(u)} = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{srec}(w)}.$$

To prove this theorem, we shall construct a new bijection $G_n \to \mathfrak{S}_n$ which will conserve the positions of records.

3.4.1 Construction of a bijection

Define $\mathcal{B}_2 : G_n \to \mathfrak{S}_n$ as the following map: $\forall\ u = b_1 \dots b_n \in G_n$ one constructs $a_1 \dots a_n \in \mathfrak{S}_n$ and introduces for later convenience an auxiliary set ϕ_u according to the following algorithm:

To begin with, one assigns $a_i = b_i$ and creates an empty set ϕ_u.
For i from 1 to n by 1
  If i appears more than once in $a_1 \cdots a_n$
    Let k be the position of the last appearance of i in $a_1 \cdots a_n$. Then $\forall j \neq k \in [n]$ such that $a_j \geq i$ increment a_j by 1.
    Add i to ϕ_u.
  EndIf
EndFor Define $a_1 \dots a_n$ as $\mathcal{B}_2(u)$.

For example:

Take $u = 31223$, $\phi_u = \{\}$. 1 appears once so we leave u unchanged. Next, 2 appears more than once. So add 1 to all the elements which are greater than or equal to 2, except the rightmost 2.

After 1 step: $u \rightsquigarrow 41324$ and $\phi_u = \{2\}$. 3 appears once so we leave u unchanged. Then, 4 appears more than once. Therefore, add 1 to all the numbers which are greater than or equal to 4, except the rightmost 4.

After 2 steps: $\mathcal{B}_2(u) = 51324$ and $\phi_u = \{2, 4\}$.

Take $u = 3411523$, $\phi_u = \{\}$.
After 1 step: $u \rightsquigarrow 4521634$ and $\phi_u = \{1\}$.
After 2 steps: $\mathcal{B}_2(u) = 5621734$ and $\phi_u = \{1, 4\}$.

Notice that $a_1 \dots a_n$ is always a permutation: After k steps, each one of the integers $1, 2, \dots, k$ will appear exactly once upon the application of the algorithm. Thus after at most n steps we will obtain a permutation.

The bijection $\mathcal{B}_2$ is similar to the Schensted's standardization map [8]. Theorem 3.10 proves that $\mathcal{B}_2(w)$ is obtained from w by reading w from right to left, by labeling $1, 2, \dots$ the successive occurrences of 1 in w, then by doing the same with the successive occurrences of 2 and so on. Standardization is the same, except that w is read from left to right instead of right to left.

3.4.2 Constructing $\mathcal{B}_2^{-1}$

In fact, it is very difficult to inverse $\mathcal{B}_2$ without doing some tricky manipulations: that is why have introduced the set ϕ_u. Knowing $\mathcal{B}_2(u)$ and ϕ_u we can easily reconstruct u: Replace every element a_i of $\mathcal{B}_2(u)$ by $a_i - \mathrm{Card}(\{j \in \phi_u | j < a_i\})$, that is substract from a_i the number of elements in ϕ_u that are less than a_i. This is only a consequence of our algorithm.

<u>For example:</u>

Take $\mathcal{B}_2(u) = 51324$ and $\phi_u = \{2, 4\}$,
$\mathrm{Card}(\{j \in \{2,4\} | j < 5\}) = 2$
$\mathrm{Card}(\{j \in \{2,4\} | j < 1\}) = 0$
$\mathrm{Card}(\{j \in \{2,4\} | j < 3\}) = 1$
$\mathrm{Card}(\{j \in \{2,4\} | j < 2\}) = 0$
$\mathrm{Card}(\{j \in \{2,4\} | j < 4\}) = 1$
So, we get: $u = (5-2)(1-0)(3-1)(2-0)(4-1) = 31223$, as in 3.4.1.
Now we have to find ϕ_u only from $\mathcal{B}_2(u)$ and the analysis becomes more complicated. However, this can be done by virtue of the following lemma:

Lemma 3.8. *The following two implications hold:*

$$k \in \phi_u \implies k+1 \text{ appears to the left of } k \text{ in } \mathcal{B}_2(u). \tag{5}$$
$$k \notin \phi_u \implies k+1 \text{ appears to the right of } k \text{ in } \mathcal{B}_2(u). \tag{6}$$

<u>For example:</u>

Take $\mathcal{B}_2(u) = 51324$.

- 2 is on the right of 1, so $1 \notin \phi_u$
- 3 is on the left of 2, so $2 \in \phi_u$
- 4 is on the right of 3, so $3 \notin \phi_u$

- 5 is on the left of 4, so $4 \in \phi_u$

So $\phi_u = \{2, 4\}$, as above.

Proof of (5) : Suppose $k \in \phi_u$. When constructing $\mathcal{B}_2(u)$, consider the step when, inside the loop **for**, $i = k$. Since $k \in \phi_u$, k appears at least twice. Moreover, by means of the algorithm, the rightmost k will give rise to k in $\mathcal{B}_2(u)$, and the second rightmost k will give rise to $k+1$ in $\mathcal{B}_2(u)$, so that $k+1$ will appear to the left of k in $\mathcal{B}_2(u)$. ■

Proof of (6) : Suppose $k \notin \phi_u$. When constructing $\mathcal{B}_2(u)$, consider the step when, inside the loop **for**, $i = k$. Suppose that $k+1$ does not appear to the right of k. Let p be the position of k (b_p will be the number that is in p-th position in u). Then, by performing backwards the steps we have just done, the numbers $b_p + 1$ will not appear to the right of b_p, which contradicts the fact that u is a good sequence. ■

Note that lemma 3.8 is equivalent to the following statement:

Lemma 3.8 bis:

$$\begin{aligned} k \in \phi_u \quad &\Leftrightarrow \quad k+1 \textit{ appears to the left of } k \textit{ in } \mathcal{B}_2(u). \\ k \notin \phi_u \quad &\Leftrightarrow \quad k+1 \textit{ appears to the right of } k \textit{ in } \mathcal{B}_2(u). \end{aligned}$$

Using the previous arguments, we can finally describe $\mathcal{B}_2^{-1}$ by the following map: $w \in \mathfrak{S}_n$. Let $\phi_w = \{j \in [n-1] \mid j+1 \text{ appears to the left of } j \text{ in } w\}$.

$$\text{For all element } a_i \text{ of } w, \text{ replace } a_i \text{ by } a_i - \mathrm{Card}(\{j \in \phi_w \mid j < a_i\}).$$

Define this new sequence as $\mathcal{B}_2^{-1}(w)$. It is clear that $\mathcal{B}_2^{-1}$ is the inverse of $\mathcal{B}_2$ by the lemma. Hence $\mathcal{B}_2$ is a bijection. ■

3.4.3 Proof of theorem 3.7

Theorem 3.7 can be easily proved by making use of the bijection $\mathcal{B}_2$.

Definition: Let $\mathrm{c}(n, k)$ be the number of elements of $\mathfrak{S}_n$ with exactly k cycles. This number is called an *unsigned Stirling number of the first kind* [3].

Proof of theorem 3.7: $u \in G_n$. Let $i, j \in [n]$ with $i \neq j$. Suppose $u = b_1 \ldots b_n$ and $\mathcal{B}_2(u) = a_1 \ldots a_n$. Clearly if $b_i > b_j$ then $a_i > a_j$ because b_i will be incremented at least the same number of times as b_j, and if $b_i = b_j$ for $i > j$ then $a_i > a_j$ because b_i will be incremented at least one more time than b_j, because b_i is to the left of b_j.

Hence $\mathcal{B}_2$ conserves records, that is, the positions of the records and their number. Hence:

$$\sum_{u \in G_n} q^{\mathrm{rec}(u)} = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{rec}(w)} \qquad \text{and} \qquad \sum_{u \in G_n} q^{\mathrm{srec}(u)} = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{srec}(w)}.$$

Lemma 3.9. *The number of elements of G_n with exactly k records is equal to $\mathrm{c}(n,k)$.*

Proof: By the preceding result, there is the same number of elements of G_n with k records as the number of elements of $\mathfrak{S}_n$ with k records. Moreover, there is a $\mathfrak{S}_n \to \mathfrak{S}_n$ bijection described in [3], which maps a permutation with k records into a permutation with k cycles and conversely (see appendix). Hence the number of elements of $\mathfrak{S}_n$ with k records is equal to $\mathrm{c}(n,k)$ and, therefore, the number of elements of G_n with k records is equal to $\mathrm{c}(n,k)$.

Moreover, it can be proved (see, e.g., [3]) that:

$$\sum_{k=0}^{n} \mathrm{c}(n,k)q^k = q(q+1)\cdots(q+n-1).$$

Finally:

$$\sum_{u\in G_n} q^{\mathrm{rec}(u)} = \sum_{w\in\mathfrak{S}_n} q^{\mathrm{rec}(w)} = \sum_{k=0}^{n} \mathrm{c}(n,k)q^k = q(q+1)\cdots(q+n-1).$$

■

3.4.4 Another interpretation of $\mathcal{B}_2$

The bijection $\mathcal{B}_2$ admits the following interpretation:

Let $w = a_1 \dots a_n \in \mathfrak{S}_n$. Consider a circular drive with n houses enumerated as $a_n, a_{n-1}, \dots, a_1$ in a clockwise order. A postman carries n letters enumerated $1, 2, \dots, n$ which must be delivered according to the following rules: (i) the letter $k = 1, \dots, n$ has to be delivered to the house k and (ii) the letter k is delivered before the letter $k+1$. He starts at the house a_n and moves clockwise. When he delivers a letter to the house a_k, he assigns to the house a_k the number b_k which is equal to the number of times he had passed in front of the house a_n.

Theorem 3.10. *We have: $b_1 \dots b_n = \mathcal{B}_2^{-1}(w)$.*

For example, take $w = 451362$. He delivers the letter 1 after the first passage. Then he delivers the letters 2, 3 and 4 after the second passage, he deliver the letter 5 after the third passage and finally he delivers the letter 6 after the fourth passage. And indeed we have $\mathcal{B}_2^{-1}(451362) = 231242$.

Proof: Call a decreasing sequence $u_1 \dots u_n$ *step-by-step decreasing* if $u_{i+1} = u_i - 1$. We will prove the following statement: Let $w = a_1 \dots a_n \in \mathfrak{S}_n$ and $i_1 < \cdots < i_l$. Assume that (i) $a_{i_1}, \dots, a_{i_l}$ is step-by-step decreasing and (ii) $a_{i_1}+1$ does not appear to the left of a_{i_1} in w. Let m satisfy $a_m = a_{i_1}+1$ (if a_{i_1} is not the greatest element of w then m exists). Let $u = b_1 \dots b_n = \mathcal{B}_2^{-1}(w)$. Then $b_{i_1} = b_{i_2} = \cdots = b_{i_l} = b_m - 1$.

Let $k = \mathrm{Card}(\{x \in \phi_u \mid x < a_{i_l}\})$, that is the number of elements of ϕ_u that are less than a_{i_l}. Then by definition of $\mathcal{B}^{-1}$, $b_{i_l} = a_{i_l} - k$. Since $i_1 < \cdots < i_l$ and since $a_{i_1}, \dots, a_{i_l}$ is step-by-step decreasing, $\forall\ j \in \mathbb{N}$ such that $2 \le j \le l$ we have $a_{i_j} \in \phi_u$. Hence $\mathrm{Card}(\{x \in \phi_u \mid x < a_{i_{l-1}} = a_{i_l}+1\}) = k+1$ and $b_{i_{l-1}} = a_{i_{l-1}} - (k+1) = a_{i_l}+1-(k+1) = a_{i_l} - k = b_{i_l}$. By induction, $\forall\ j \in \mathbb{N}$ such that $2 \le j \le l$, $b_{i_j} = a_{i_l} - k = b_{i_l}$. Moreover, since $a_{i_1}+1$ does not appear to the left of a_{i_1}

in w, we have $a_{i_1} \notin \phi_u$. Hence $\mathrm{Card}(\{x \in \phi_u \mid x < a_m = a_{i_1} + 1\} = \mathrm{Card}(\{x \in \phi_u \mid x < a_{i_1}\}$. Thus:

$$b_m = a_m - \mathrm{Card}(\{x \in \phi_u \mid x < a_{i_1}\} = 1 + a_{i_1} - (k+l-1) = 1 + a_{i_l} + (l-1) - (k+l-1) = 1 + a_{i_l} - k = 1 + b_{i_l}.$$

So $b_m - 1 = b_{i_l}$ as announced.

To prove theorem 3.10, let w_1 be the set (*listed in decreasing order*) of the positions of the elements of the largest step-by-step decreasing subsequence with the least element 1; w_2 be the set (*listed in decreasing order*) of the positions of the elements of the largest step-by-step decreasing subsequence with the least element $\max(w_1)+1, \ldots$; w_m be set (*listed in decreasing order*) of the positions of the elements of the largest step-by-step decreasing subsequence with least element $\max(w_{m-1}) + 1$ and such that $\max(w_m) = \max(w)$. By the preceding statement, there exists k such that:

$$\begin{array}{rcll}
b_i & = & k & \text{for } i \in w_1 \\
b_i & = & k+1 & \text{for } i \in w_2 \\
 & \vdots & & \\
b_i & = & k+j-1 & \text{for } i \in w_j \\
 & \vdots & & \\
b_i & = & k+m-1 & \text{for } i \in w_m.
\end{array} \tag{7}$$

By lemma 2.3, $k = 1$ and theorem 3.10 is proved. Note also that $\max(u) = m$. ■

For example: Take $w = 3562417$. One has :

- $w_1 = \{6, 4, 1\}$, since the largest step-by-step decreasing subsequence with the least element 1 is $\boxed{3}56\boxed{2}4\boxed{1}7$.
- $w_2 = \{5, 2\}$, since the largest step-by-step decreasing subsequence with the least element 4 is $3\boxed{5}62\boxed{4}17$.
- $w_3 = \{3\}$, since the largest step-by-step decreasing subsequence with the least element 6 is $35\boxed{6}2417$.
- $w_4 = \{7\}$, since the largest step-by-step decreasing subsequence with the least element 7 is $356241\boxed{7}$.

And indeed, since $u := b_1 \ldots b_7 := \mathcal{B}_2^{-1}(3562417) = 1231214$,

$$\begin{array}{rcll}
b_i & = & 1 & \text{for } i \in \{6, 4, 1\} \\
b_i & = & 2 & \text{for } i \in \{5, 2\} \\
b_i & = & 3 & \text{for } i \in \{3\} \\
b_i & = & 4 & \text{for } i \in \{7\}.
\end{array}$$

3.4.5 Properties of ϕ_u

In this section we will study the properties of ϕ_u in more details.

Definitions: Let $w \in \mathfrak{S}_n$.

- Recall that $\mathrm{des}(w)$ is the *descent set* of w, that is, the set of all descents of w.
- Let $\mathrm{i}(w)$ be the *inverse permutation* of w (it is usually denoted by w^{-1}, but as we will see below, it is more convenient to name it $\mathrm{i}(w)$). By definition of $\mathrm{i}(w)$, if $\mathrm{i}(w) = a_1 \dots a_n$ then a_i is the position of i in w.

Theorem 3.11. *Let $u \in G_n$. Then ϕ_u is the descent set of $i(\mathcal{B}_2(u))$.*

Proof: Let $u = b_1 \dots b_n \in G_n$ and $\mathcal{B}_2(u) = a_1 \dots a_n$. By lemma 3.8:

$$\begin{aligned} k \in \phi_u \quad &\Leftrightarrow \quad k+1 \text{ appears to the left of } k \text{ in } \mathcal{B}_2(u) \\ &\Leftrightarrow \quad \mathrm{i}(\mathcal{B}_2(u))_k > \mathrm{i}(\mathcal{B}_2(u))_{k+1} \\ &\Leftrightarrow \quad k \text{ is a descent in } i(\mathcal{B}_2(u)), \end{aligned}$$

where $\mathrm{i}(\mathcal{B}_2(u))_k$ is the k-th element of $\mathrm{i}(\mathcal{B}_2(u))$. ■

As in the proof of theorem 3.10, we introduce the sets $w_1, w_2, \dots, w_m$ for a permutation w.

Lemma 3.12. *We have: $i(w) = w_1 w_2 \dots w_m$.*

Proof: The elements of w with positions corresponding to the elements of w_1 form a step-by-step decreasing subsequence. Hence the positions of $1, 2, \dots, \mathrm{Card}(w_1)$ in w are exactly the elements of w_1. By the same argument, the positions of $\mathrm{Card}(w_1)+1$, $\mathrm{Card}(w_1)+2, \dots, \mathrm{Card}(w_1)+\mathrm{Card}(w_2)$ in w are exactly the elements of w_2 and so on. Hence $\mathrm{i}(w) = w_1 w_2 \dots w_m$.
Note that we identify the set w_1 to the sequence obtained by writing down one-by-one the elements of w_1, starting by the first one. ■

For example, for $w = 3562417$ one has $w_1 = \{6, 4, 1\}$, $w_1 = \{5, 1\}$, $w_1 = \{3\}$, $w_1 = \{7\}$ and:

$$\mathrm{i}(3562417) = 6415237 = 641\,52\,3\,7 = w_1 w_2 w_3 w_4.$$

Definitions: Let $w \in \mathfrak{S}_n$.

- An *ascent* of w is a number i for which $a_{i+1} > a_i$. $\mathrm{asc}(w)$ is the *ascent set* of w, that is, the set of all ascents of w. E.g, $\mathrm{asc}(\underline{4}75\underline{1}\underline{6}8\underline{2}3) = \{1, 4, 5, 7\}$.
- Let $\mathrm{r}(w)$ be the *reverse permutation* of w (if $w = a_1 a_2 \dots a_n$ then $\mathrm{r}(w) = a_n \dots a_2 a_1$).

Theorem 3.13. *The two following relations hold:*

$$Card(\phi_u) = n - max(u) \qquad and \qquad \sum_{u \in G_n} q^{Card(\phi_u)} = \frac{A_n(q)}{q}.$$

Proof: Let $u = b_1 \dots b_n$, $\mathcal{B}_2(u) = w = a_1 \dots a_n$ and $\max(u) = k+1$. By lemma 3.12, split $\mathrm{i}(w)$ into $k+1$ *subsequences* $w_1, \dots, w_{k+1}$ such that each w_i does not contain an ascent, i.e., w_i is decreasing and such that $\mathrm{i}(w) = w_1 \dots w_{k+1}$. It follows that $\mathrm{Card}(\mathrm{asc}(\mathrm{i}(w))) = k$. Since the reverse permutation bijection maps an ascent to a descent and conversely, we obtain:

$$\mathrm{Card}(\mathrm{asc}(\mathrm{i}(w))) + \mathrm{Card}(\mathrm{des}(\mathrm{i}(w))) = n - 1.$$

By theorem 3.11, ϕ_u is the descent set of $\mathrm{i}(w)$. Hence

$$k + \mathrm{Card}(\phi_u) = n - 1$$

or

$$\mathrm{Card}(\phi_u) = n - (k+1).$$

Finally,

$$\mathrm{Card}(\phi_u) = n - \max(u).$$

Therefore, we obtain:

$$\sum_{u \in G_n} q^{\mathrm{Card}(\phi_u)} = \sum_{u \in G_n} q^{n - \max(u)} = \frac{q^n}{\sum_{u \in G_n} q^{\max(u)}} = q^n \sum_{u \in G_n} \frac{1}{q^{\max(u)}}.$$

The idea is to compute the coefficient of q^j in the rightmost expression. This can be done owing to the following lemma:

Lemma 3.14. *The Eulerian polynomial is symmetric, that is, $A(n,k) = A(n, n+1-k)$.*

Proof: It suffices to show that in $\mathfrak{S}_n$ the number of permutations with $k-1$ descents is equal to the number of permutations with $n-k$ descents, which is true since the reverse bijection maps a permutation with $k-1$ descents to a permutation with $n-1-(k-1) = n-k$ descents. ■

Let $Q = q^n \sum_{u \in G_n} \frac{1}{q^{\max(u)}}$.

By theorem 3.2 the number of elements of G_n with the greatest element k is equal to $A(n,k)$. Hence the coefficient of q^{n-i} in Q is $A(n,i)$. Let $j = n-i$, so that the coefficient of q^j in Q is $A(n, n-j)$. By lemma 3.14, the coefficient of q^j in Q is $A(n, j+1)$. Hence:

$$\sum_{u \in G_n} q^{\mathrm{Card}(\phi_u)} = \frac{A_n(q)}{q}.$$

■

Note that (7) is very similar to (4). This suggests that the set ϕ_u is closely related to the bijection $\mathcal{B}^{-1}$.

Theorem 3.15. *Let $w \in \mathfrak{S}_n$. Then $\mathcal{B}_2(\mathcal{B}^{-1}(w)) = i(r(w))$.*

Proof: For all $i \in [k+1]$ let $w'_i = \mathrm{r}(w_i)$. Using the same definitions as in the proof of theorem 3.13, $\mathrm{r}(\mathrm{i}(w)) = w'_{k+1} \dots w'_1$. Note that since w_i is decreasing, w'_i is increasing and $r(i(w))$ has exactly k descents. By (7):

$$\begin{aligned} b_i &= 1 \quad \text{for } i \in w'_1 \\ b_i &= 2 \quad \text{for } i \in w'_2 \\ &\vdots \\ b_i &= j \quad \text{for } i \in w'_j \\ &\vdots \\ b_i &= k+1 \quad \text{for } i \in w'_{k+1}. \end{aligned} \tag{8}$$

Note that here we also identify the sequence w'_j to the set of all the elements of the sequence w'_j. After comparing (8) and (4) one immediately finds that $\mathcal{B}^{-1}(\mathrm{r}(\mathrm{i}(w))) = b_1 \dots b_n = u$. Thus:

$$\mathcal{B}^{-1}(\mathrm{r}(\mathrm{i}(\mathcal{B}_2(u)))) = u$$

and $\mathcal{B}_2(u) = \mathrm{i}(\mathrm{r}(\mathcal{B}(u)))$. Take w' such that $u = \mathcal{B}^{-1}(w')$. Then $\mathcal{B}_2(\mathcal{B}^{-1}(w')) = \mathrm{i}(\mathrm{r}(w'))$. ■

For example: For $w = 3562417$, one has $w'_1 = 146$, $w'_2 = 25$, $w'_3 = 3$, $w'_4 = 7$. First,

$$\mathrm{r}(\mathrm{i}(w)) = 7325146 = 7\,3\,25\,146 = w'_4 w'_3 w'_2 w'_1.$$

Second, $u := b_1 \dots b_7 := \mathcal{B}_2^{-1}(3562417) = 1231214$ and indeed one has:

$$\begin{aligned} b_i &= 1 \quad \text{for } i \in \{1,4,6\} \\ b_i &= 2 \quad \text{for } i \in \{2,5\} \\ b_i &= 3 \quad \text{for } i \in \{3\} \\ b_i &= 4 \quad \text{for } i \in \{7\}. \end{aligned}$$

3.5 Avoiding patterns

Definitions:

- Let $w = a_1 \cdots a_k \in \mathfrak{S}_k$. We say that the sequence $b_1 \dots b_n \in \mathfrak{S}_n$ (with $k \leq n$) *avoids* w if none of its subsequences of length k is ordered in "the same way" as w, that is, there exist no k integers $i_1 < i_2 < \cdots < i_k$ such that $b_{i_r} < b_{i_s} \iff a_r < a_s$. For instance, $\underline{5}21\underline{34}$ does not avoid 312 because 534 is ordered in the same way as 312, but 52134 is 2413 avoiding.

- Define a *full sequence* s to be a sequence of k positive integers with the greatest element m such that all $i \in [m]$ appear in s. Denote by F_k the set of *full sequences* of length k. Let u be a good sequence of length $n \geq k$. Similarly to above, we can introduce the following definition: $u = b_1 \dots b_n$ *avoids a full sequence* $s = a_1 \dots a_k$ if there are no k integers $i_1 < i_2 < \cdots < i_k$ such that $b_{i_r} < b_{i_s} \iff a_r < a_s$ and $b_{i_r} = b_{i_s} \iff a_r = a_s$. For example, $u = \underline{13}2\underline{34}22$ does not avoid $s = 1223$ but u avoids $s = 13324$.

Suppose that $w \in \mathfrak{S}_n$ avoids a permutation. The goal of this section is to find what kind of sequences does $\mathcal{B}_2^{-1}(w)$ avoid.

Let $a_1 \dots a_n \in \mathfrak{S}_n$. Define the set $\mathfrak{M}_{a_1 \dots a_n}$ as follows:

$$\mathfrak{M}_{a_1 \dots a_n} = \{s_1 \dots s_n \in F_n \mid \text{ for } i < j \ , (a_i < a_j) \Longleftrightarrow (s_i < s_j)\}. \tag{9}$$

For instance, for $w = 4132$, $\mathfrak{M}_{4132} = \{2122, 3122, 3132, 4132\}$.

Theorem 3.16. *Let $k \leq n$ and $w \in \mathfrak{S}_k$. The number of permutations of $[n]$ avoiding w is equal to the number of good sequences of length n avoiding simultaneously all elements of $\mathfrak{M}_w$.*

Proof: Let $w = a_1 \dots a_n \in \mathfrak{S}_n$, $u = \mathcal{B}_2^{-1}(w) = b_1 \dots b_n$ and $w = c_1 \dots c_k \in \mathfrak{S}_k$. We will prove the following statement:

$$\exists\, x \in \mathfrak{M}_w \text{ such that } u \text{ does not avoid } x \Longleftrightarrow w \text{ does not avoid } w.$$

First, suppose that $\exists\, x \in \mathfrak{M}_w$ such that u does not avoid $x = d_1 \dots d_k$. There exists $i_1 < \cdots < i_k$ such that $b_{i_1} b_{i_2} \dots b_{i_k}$ is ordered in the same way as x. Assume that $l < m$. By definition of x, $c_l < c_m \Longleftrightarrow d_l < d_m \Longleftrightarrow b_{i_l} < b_{i_m}$. By definition of $\mathcal{B}_2$, $b_{i_l} < b_{i_m} \Longleftrightarrow a_{i_l} < a_{i_m}$. Hence $c_l < c_m \Longleftrightarrow a_{i_l} < a_{i_m}$ and $a_{i_1} a_{i_2} \dots a_{i_k}$ is ordered in the same way as w, which implies that w does not avoid w.

Second, suppose that w does not avoid w. There exists $i_1 < \cdots < i_k$ such that $a_{i_1} a_{i_2} \dots a_{i_k}$ is ordered in the same way as w. Assume that $l < m$. Then $c_l < c_m \Longleftrightarrow a_{i_l} < a_{i_m}$. By definition of $\mathcal{B}_2^{-1}$, $a_{i_l} < a_{i_m} \Longleftrightarrow b_{i_l} < b_{i_m}$. Let $b = b_{i_1} b_{i_2} \dots b_{i_k}$. Take[2]:

$$x = \overline{(b_{i_1} - (\min(b) - 1))(b_{i_2} - (\min(b) - 1)) \dots (b_{i_k} - (\min(b) - 1))}\ .$$

Since:

$$c_l < c_m \Longleftrightarrow a_{i_l} < a_{i_m} \Longleftrightarrow b_{i_l} < b_{i_m} \Longleftrightarrow x_l < x_m,$$

we have $x \in \mathfrak{M}_w$ and the proof is finished. ■

We extend $\mathcal{B}_2$ on F_n by calculating $\mathcal{B}_2(s)$ for each element $s \in F_n$ by applying the definition of $\mathcal{B}_2$ (from section 3.4.1). The second part of the proof of theorem 3.16 yields the following corollary:

Corollary 3.17. *Let $w \in \mathfrak{S}_n$. Then $\mathfrak{M}_w = \{s \in F_n \mid \mathcal{B}_2(s) = w\}$.*

Define $C_n = \frac{1}{n+1}\binom{2n}{n}$ as the n-th Catalan number.

Corollary 3.18. *The number of good sequences of length n which avoid 123 is equal to C_n.*

Proof: It is known that the number of permutations of length n which avoid 123 is equal to C_n [4]. It is clear that $\mathfrak{M}_{123} = \{123\}$. By theorem 3.16, the number of permutations of $[n]$ avoiding 123 is equal to the number of good sequences of length n avoiding 123. ■

For $w \in \mathfrak{S}_n$, define ϕ_w as the set $\phi_{\mathcal{B}_2^{-1}(w)}$. Recall that des$(w)$ is the descent set of w.

[2]We take such an x in order to obtain a full sequence when $\min(b) \neq 1$

Theorem 3.19. *The following property holds:* $Card(\mathfrak{M}_w) = 2^{Card(des(i(w)))}$.

For example:

- $w = 123$ and $\mathfrak{M}_{123} = \{123\}$. Card(des(i(123))) = Card(des(123)) = Card($\emptyset$) = 0.
- $w = 132$ and $\mathfrak{M}_{132} = \{122, 132\}$. Card(des(i(132))) = Card(des(132)) = Card($\{2\}$) = 1.
- $w = 213$ and $\mathfrak{M}_{213} = \{112, 213\}$. Card(des(i(213))) = Card(des(213)) = Card($\{1\}$) = 1.
- $w = 231$ and $\mathfrak{M}_{231} = \{121, 231\}$. Card(des(i(231))) = Card(des(312)) = Card($\{1\}$) = 1.
- $w = 312$ and $\mathfrak{M}_{312} = \{212, 312\}$. Card(des(i(312))) = Card(des(231)) = Card($\{2\}$) = 1.
- $w = 321$ and $\mathfrak{M}_{321} = \{111, 211, 221, 321\}$. Card(des(i(321))) = Card(des(321)) = Card($\{1, 2\}$) = 2.

Lemma 3.20. *Let* $w = a_1 \dots a_n \in \mathfrak{S}_n$ *and* $s = s_1 \dots s_n \in \mathfrak{M}_w$. *Assume that* k, l *are positive integers such that* $a_k < a_l$. *Then* $s_k \leq s_l$.

Proof: If $k < l$, then by definition of s, $a_k < a_l \Longrightarrow s_k < s_l \Longrightarrow s_k \leq s_l$. Now assume $k > l$ and suppose that $s_k > s_l$. By definition of s, since $l < k$, $s_l < s_k \Longrightarrow a_l < a_k$ which is a contradiction. ■

Proof of theorem 3.19: Let $w = a_1 \dots a_n \in \mathfrak{S}_n$ and let $s = s_1 \dots s_n \in \mathfrak{M}_w$. Assume that k, l are positive integers such that $a_l = a_k + 1$. Clearly if $s_l \geq s_k + 2$ then $s_k + 1$ does not appear in s by lemma 4.1, which contradicts the fact that s is a good sequence.

First, suppose that $a_k \in \phi_w$. By lemma 4.1, $s_l = s_k + 1$ or $s_l = s_k$. Let $j = a_k$ and let $\mathrm{i}(w) = i_1 \dots i_n$. Since i_j is the position of j in w, $k = i_j$ and $l = i_{j+1}$. Hence:

$$j \in \phi_w \Longrightarrow s_{i_{j+1}} = s_{i_j} \text{ ou } s_{i_{j+1}} = s_{i_j} + 1. \tag{10}$$

Second, suppose that $a_k \notin \phi_w$, that is $k < l$. By definition of s, $a_k < a_l \Longrightarrow s_k < s_l \Longrightarrow s_l = s_k + 1$ by lemma 4.1. Similarly, let $j = a_k$ and let $\mathrm{i}(w) = i_1 \dots i_n$ so that $k = i_j$ and $l = i_{j+1}$. Hence:

$$j \notin \phi_w \Longrightarrow s_{i_{j+1}} = s_{i_j} + 1. \tag{11}$$

We shall now give an algorithm which will produce $2^{\mathrm{Card}(\phi_w)}$ elements of $\mathfrak{M}_w$:

Let $\mathrm{i}(w) = i_1 \dots i_n$. Recall that i_j is the position of j in w. Start with the set $\mathbb{S}$, where $\mathbb{S} = \{11 \dots 11\}$ contains the sequence of n 1s.

For j from 1 to $n - 1$ by 1
  If $j \in \phi_w$
    $*$ For each existing sequence $s = s_1 \dots s_n$ in $\mathbb{S}$, one assigns $s_{i_{j+1}} = s_{i_j}$
    $*$ One creates a new sequence $s' = s'_1 \dots s'_n$ such that $s' = s$, assigns $s'_{i_{j+1}} = s'_{i_j} + 1$ and adds s' to $\mathbb{S}$.
  Else ($j \notin \phi_w$)
    For each existing sequence $s = s_1 \dots s_n$ in $\mathbb{S}$, one assigns $s_{i_{j+1}} = s_{i_j} + 1$.
  EndIf
EndFor

Define $\mathbb{S}_w$ to be the final set $\mathbb{S}$.

For example for $w = 25143$. $\phi_{25143} = \{1, 3, 4\}$. Start with $\mathbb{S} = \{11111\}$
For $j = 1$, since $1 \in \phi_{25143}$, we obtain $\mathbb{S} = \{11111\ ,\ 21111\}$.
For $j = 2$, since $2 \notin \phi_{25143}$, we obtain $\mathbb{S} = \{11112, 21113\}$.
For $j = 3$, since $3 \in \phi_{25143}$, we obtain $\mathbb{S} = \{11122, 11132\ ,\ 21133, 21143\}$.
For $j = 4$, since $4 \in \phi_{25143}$, we finally obtain
$\mathbb{S}_{25143} = \{12122, 13122\ ,\ 13132, 14132\ ,\ 23133, 24133\ ,\ 24143, 25143\}$.

Proposition 3.21. *The following inclusion is verified:* $\mathfrak{M}_w \subset \mathbb{S}_w$.

Proof: This inclusion is obvious because every element of $\mathfrak{M}_w$ verifies (10) and (11), so that it will be produced by the algorithm and will therefore be an element of $\mathbb{S}_w$. ■

Proposition 3.22. *The following inclusion is verified:* $\mathbb{S}_w \subset \mathfrak{M}_w$.

Proof: Let $w = a_1 \ldots a_n$ be a fixed permutation and let $s \in \mathbb{S}_w$. Let $\mathrm{i}(w) = i_1 \ldots i_n$. By construction, s is a full sequence. We show that for $x < y$, $(a_x < a_y) \Longleftrightarrow (s_x < s_y)$.

Let x, y be such that $x < y$ and $a_x < a_y$. Thus there exists such a k that $x \leq k < y$ and $a_k \notin \phi_w$. The algorithm shows that:

$$s_x \leq s_{x+1} \leq \cdots \leq s_k < s_{k+1} \leq \cdots \leq s_y,$$

so that $s_x < s_y$.

Let x, y be such that $x < y$ and $s_x < s_y$. By construction, if $s_{i_k} < s_{i_l}$ then $k < l$. But $s_{i_{a_x}} < s_{i_{a_y}}$ and $x = i_{a_x}$. Hence $a_x < a_y$. ■

Hence we deduce that $\mathbb{S}_w$ is exactly $\mathfrak{M}_w$. The number of elements of $\mathbb{S}_w$ is $2^{\mathrm{Card}(\phi_w)}$. Moreover, by theorem 3.11: $\mathrm{Card}(\phi_w) = \mathrm{Card}(\mathrm{des}(\mathrm{i}(w))))$. We conclude that:

$$\mathrm{Card}(\mathfrak{M}_w) = 2^{\mathrm{Card}(\phi_w)} = 2^{\mathrm{Card}(\mathrm{des}(\mathrm{i}(w)))}.$$

■

Let $f(n)$ be the number of ways n competitors can rank in a competition, allowing for the possibility of ties [2]. For instance, $f(3) = 13$ (six ways with no ties, three ways with a two-way tie for the first place, three ways with a two-way tie for the second place, and one way with all three tied).

It is easy to see that $f(n) = \mathrm{Card}(F_n)$, since a such ranking can be put in a bijection with a full sequence $s_1 \ldots s_n$ in the following way: s_i is the position of i in the ranking.

Corollary 3.23. *We have:*

$$f(n) = \frac{A_n(2)}{2}.$$

Proof: Consider all permutations $w_1 \dots w_{n!}$ of $[n]$. The sets $\mathfrak{M}_{w_1} \dots \mathfrak{M}_{w_{n!}}$ are clearly disjoint[3] so that $F_n = \mathfrak{M}_{w_1} \bigcup \cdots \bigcup \mathfrak{M}_{w_{n!}}$. Let w be a permutation such that $\mathrm{i}(w)$ has k descents. By theorem 3.19, $\mathrm{Card}(\mathfrak{M}_w) = 2^k$. But the number of the permutations of $[n]$ with k descents is $A(n, k+1)$. Hence:

$$f(n) = \sum_{k=0}^{n-1} 2^k A(n, k+1) = \frac{\sum_{k=0}^{n-1} 2^{k+1} A(n, k+1)}{2} = \frac{A_n(2)}{2}.$$

■

Note: It can be proved (see Ref. [2]) that the exponential generating function of $f(n)$ is $\dfrac{1}{2 - e^x}$.

4 Application to permutations

4.1 Reminder

We have constructed two bijections:

$$\mathcal{B} : G_n \to \mathfrak{S}_n$$

and

$$\mathcal{B}_2 : G_n \to \mathfrak{S}_n.$$

We have found that $\mathcal{B}_2 \circ \mathcal{B}^{-1}$ is endowed with an interesting interpretation by virtue of theorem 3.15. Let $\mathcal{B}_1 = \mathcal{B}^{-1}$ and $\mathcal{B}_3 = \mathcal{B}_2 \circ \mathcal{B}_1$. Thus $\mathcal{B}_3$ is a $\mathfrak{S}_n \to \mathfrak{S}_n$ bijection. The following question naturally arises:

Does $\mathcal{B}_3$ have distinguished properties?

The answer to this question is yes.

4.2 $\mathcal{B}_3$ and inversions

4.2.1 Simplifying $\mathcal{B}_3$

First let us give yet another proof of theorem 3.15. The latter states that if $w \in \mathfrak{S}_n$ then $\mathcal{B}_3(w) = \mathrm{i}(\mathrm{r}(w))$.

Lemma 4.1. *Let $w = a_1 \dots a_n \in \mathfrak{S}_n$ and $\mathcal{B}_3(w) = b_1 \dots b_n$. Then $b_{a_1} = n$, $b_{a_2} = n-1$, $\dots$, $b_{a_n} = 1$. In other words, $\mathcal{B}_3(w)$ is obtained from w by placing n in the a_1-th position, $n-1$ in the a_2-th position , $\dots$, 1 in the a_n-th position.*

Proof: As in (3.1), suppose that w has exactly m descents. One can naturally split w into $m+1$ *subsequences* $w_1, \dots, w_{m+1}$ such that each w_i does not contain a descent, i.e., w_i is increasing. In $\mathcal{B}_1(w)$, all the positions that are elements of w_1 will be occupied by $m+1$ according to (4). What is the result of applying $\mathcal{B}_2$ to $\mathcal{B}_1(w)$?

We have already seen that if elements are equal, then after applying $\mathcal{B}_2$, the leftmost element will be the largest one, the second leftmost element will be the second largest one, etc. Since

[3] Suppose that $s = s_1 \dots s_n \in \mathfrak{M}_{a_1 \dots a_n}$ and that $s' = s'_1 \dots s'_n \in \mathfrak{M}_{a_1 \dots a_n}$. By definition (9), for $i < j$ one has $s_i < s_j \iff s'_i < s'_j$. Since s and s' are permutations, $s = s'$.

$\mathcal{B}_3(w)$ is a permutation, the leftmost position of w_1, that is, the position a_1, will be occupied by n, a_2 will be occupied by $n-1$ and so on. ■

Second proof of theorem 3.15: Suppose that $w = a_1 \dots a_n \in \mathfrak{S}_n$. By lemma **??**, $\mathcal{B}_3(w)$ is obtained from w by placing n in the a_1-th position, $n-1$ in the a_2-th position , ..., 1 in the a_n-th position. Hence $\mathcal{B}_3(\mathrm{r}(w))$ is obtained from w by placing n in the a_n-th position, $n-1$ in the a_{n-1}-th position, ..., 1 in the a_1-th position. Or in other words: $\mathcal{B}_3(r(w))$ is obtained from w by placing 1 in the a_1-th position, 2 in the a_2-th position ,..., n in the a_n-th position. This procedure is precisely the construction of the inverse permutation of w! Hence $\mathcal{B}_3(\mathrm{r}(w)) = \mathrm{i}(w)$, $\mathcal{B}_3(\mathrm{r}(\mathrm{r}(w)) = \mathrm{i}(\mathrm{r}(w))$ and finally $\mathcal{B}_3(w) = \mathrm{i}(\mathrm{r}(w))$. ■

4.2.2 $\mathcal{B}_3$ and inversions

Definition: An *inversion* of $w = a_1 \dots a_n \in \mathfrak{S}_n$ is a pair (i, j) for which $i < j$ and $a_i > a_j$. $\mathrm{inv}(w)$ is *the number of inversions* of w. E.g., in 3241 there are 4 inversions: $(1, 2)$, $(1, 4)$, $(2, 4)$ and $(3, 4)$.

Due to theorem 3.15, we are now able to prove a peculiar theorem which follows:

Theorem 4.2. *Let $w \in \mathfrak{S}_n$. Then:*

$$inv(w) + inv(\mathcal{B}_3(w)) = \frac{n(n-1)}{2}.$$

Lemma 4.3. *Let $w \in \mathfrak{S}_n$. Then:*

$$inv(w) = inv(i(w)).$$

Proof: Let $w = a_1 \dots a_n$ and let us construct $\mathrm{i}(w)$ by the double array representation which we shall denote as $\mathrm{D}(w)$. Recall that the double array representation of a permutation is obtained by writing down the integers $1, 2, \dots, n$ in the first row and the permutation in the second row. To inverse a permutation w, one writes the double array representation of w, interchanges the first and second rows, orders the first row such that it is an increasing sequence, and finally reads $\mathrm{i}(w)$ in the second row.

For example let us invert $w = 456132$.

$$\mathrm{D}(w) \quad = \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 3 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 4 & 5 & 6 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

After ordering the first row, one obtains $\mathrm{D}(i(w))$:

$$\mathrm{D}(\mathrm{i}(w)) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 5 & 1 & 2 & 3 \end{pmatrix}.$$

Hence $\mathrm{i}(456132) = 465123$.

Let $w = a_1 \dots a_n$. Then:

$$\mathrm{D}(w) \quad = \quad \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & j & \cdots & n \\ a_1 & a_2 & \cdots & a_i & \cdots & a_j & \cdots & a_n \end{pmatrix}.$$

where we assumed that $i < j$. Interchanging the first and second rows, we find

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_i & \cdots & a_j & \cdots & a_n \\ 1 & 2 & \cdots & i & \cdots & j & \cdots & n \end{pmatrix}.$$

If $a_i < a_j$, i.e., (i, j) is not an inversion, we get:

$$\mathrm{D}(\mathrm{i}(w)) \quad = \quad \begin{pmatrix} a_x & \cdots & a_i & \cdots & a_j & \cdots & a_y \\ x & \cdots & i & \cdots & j & \cdots & y \end{pmatrix}.$$

This implies that in $\mathrm{i}(w)$ the pair (a_i, a_j) is not an inversion, since $i < j$.

Now suppose $a_i > a_j$, that is (i, j) is an inversion. In this case we get:

$$\mathrm{D}(\mathrm{i}(w)) \quad = \quad \begin{pmatrix} a_x & \cdots & a_j & \cdots & a_i & \cdots & a_y \\ x & \cdots & j & \cdots & i & \cdots & y \end{pmatrix}.$$

This means that in $\mathrm{i}(w)$ the pair (a_j, a_i) is an inversion, since $i < j$.

Thus, an inversion in w maps to an inversion in $\mathrm{i}(w)$ and a pair which is not an inversion in w maps to a pair which is not an inversion in $\mathrm{i}(w)$. Hence the lemma is proved. ∎

Lemma 4.4. *We have:*

$$inv(w) + inv(r(w)) = \frac{n(n-1)}{2}.$$

Proof: By definition of $\mathrm{r}(w)$, a pair (i, j), with $i > j$, which is an inversion in w will map to a pair that is not an inversion in $\mathrm{r}(w)$ and a pair (i, j), with $i > j$, which is not an inversion in w will map to a pair that is an inversion in $\mathrm{r}(w)$. Hence:

$$\mathrm{inv}(w) + \mathrm{inv}(\mathrm{r}(w)) = \binom{n}{2} = \frac{n(n-1)}{2}.$$

∎

Proof of theorem 4.2: By lemmas 3.14, 4.1 and by theorem 3.15:

$$\frac{n(n-1)}{2} = \mathrm{inv}(w) + \mathrm{inv}(\mathrm{r}(w)) = \mathrm{inv}(w) + \mathrm{inv}(\mathrm{i}(\mathrm{r}(w))) = \mathrm{inv}(w) + \mathrm{inv}(\mathcal{B}_3(w)).$$

∎

Corollary 4.5. *The following relation holds:*

$$\sum_{w \in \mathfrak{S}_n} inv(w) = \frac{n(n-1)n!}{4}.$$

Proof:

$$\sum_{w \in \mathfrak{S}_n} \frac{n(n-1)}{2} = \sum_{w \in \mathfrak{S}_n} \mathrm{inv}(w) + \mathrm{inv}(\mathcal{B}_3(w)) = \sum_{w \in \mathfrak{S}_n} \mathrm{inv}(w) + \sum_{w \in \mathfrak{S}_n} \mathrm{inv}(\mathcal{B}_3(w)) = 2 \sum_{w \in \mathfrak{S}_n} \mathrm{inv}(w).$$

Hence:

$$2\sum_{w\in\mathfrak{S}_n}\mathrm{inv}(w)=n!\times\frac{n(n-1)}{2}.$$

■

4.3 $\mathcal{B}_3$, ascents and descents

Definition : Let $\mathrm{s}(w)$ be the permutation obtained from $w=a_1\dots a_n$ by substituting each a_i by $n+1-a_i$. We shall call $\mathrm{s}(w)$ the *symmetric permutation* of w. E.g, $\mathrm{s}(47516823)=52483176$.

Theorem 4.6. *We have:*

$$asc(w)=des(\mathcal{B}_3(i(w))).$$

Lemma 4.7. *Let $w=a_1\dots a_n\in\mathfrak{S}_n$. Then $\mathcal{B}_3(i(w))$ is a symmetric permutation of w.*

Proof: The idea is to take $w'=\mathrm{i}(w)=a_1\dots a_n\in\mathfrak{S}_n$ and compare $\mathrm{i}(\mathrm{r}(w'))$ and $\mathrm{i}(w')$. As before, let us use the double array representation:

$$\mathrm{D}(w')=\begin{pmatrix}1&2&\cdots&n\\a_1&a_2&\cdots&a_n\end{pmatrix}.$$

$$\mathrm{D}(\mathrm{r}(w'))=\begin{pmatrix}1&2&\cdots&n\\a_n&a_{n-1}&\cdots&a_n\end{pmatrix}.$$

Interchanging the first and second rows in $\mathrm{D}(\mathrm{r}(w'))$, we get:

$$\begin{pmatrix}a_n&a_{n-1}&\cdots&a_1\\1&2&\cdots&n\end{pmatrix}.$$

Hence in $\mathrm{i}(\mathrm{r}(w'))$ every a_i is paired with $n+1-i$.

Moreover, after interchanging the first and second row in $\mathrm{D}(w')$, we obtain:

$$\begin{pmatrix}a_1&a_2&\cdots&a_n\\1&2&\cdots&n\end{pmatrix}.$$

Hence in $\mathrm{i}(w')$ every a_i is paired with i. Since $n+1-i+i=n+1$, $\mathrm{i}(\mathrm{r}(w'))$ and $\mathrm{i}(w')$ are symmetric. Since $w'=\mathrm{i}(w)$, $\mathrm{i}(\mathrm{r}(\mathrm{i}(w)))$ and $\mathrm{i}(\mathrm{i}(w))$ are symmetric. Thus $\mathcal{B}_3(\mathrm{i}(w))$ and w are symmetric. ■

Proof of theorem 4.6: By lemma 4.7, the bijection $w\mapsto\mathcal{B}_3(\mathrm{i}(w))$ maps every descent to an ascent and converse is also true. ■

Corollary 4.8. $\mathfrak{S}_n$ *possesses the same number of ascents and of descents:*

$$\sum_{w\in\mathfrak{S}_n} \mathrm{card}(\mathrm{asc}(w)) = \sum_{w\in\mathfrak{S}_n} \mathrm{card}(\mathrm{des}(w)) = \frac{(n-1)n!}{2}.$$

Proof: By theorem 4.6:

$$\sum_{w\in\mathfrak{S}_n} \mathrm{card}(\mathrm{asc}(w)) = \sum_{w\in\mathfrak{S}_n} \mathrm{des}(\mathcal{B}_3(\mathrm{i}(w)) = \sum_{w\in\mathfrak{S}_n} \mathrm{card}(\mathrm{des}(w)) \equiv \mathcal{N}.$$

Moreover, there are $n!$ permutations and each permutation has in total $n-1$ ascents and descents. After summing over all permutations, we get: $\mathcal{N}+\mathcal{N} = (n-1)n!$. ■

4.4 $\mathcal{B}_3$ and orbits

The goal of this section is to find the smallest integer k such that for a given w:

$$\underbrace{\mathcal{B}_3 \circ \ldots \circ \mathcal{B}_3}_{k}(w) \equiv \mathcal{B}_3^{(k)}(w) = w.$$

Theorem 4.9. *Depending on the properties of w, k takes one of the following values:*

$$k = 1 \quad \Leftrightarrow \quad r(w) = i(w) \tag{12}$$
$$k = 2 \quad \Leftrightarrow \quad r(s(w)) = w \tag{13}$$
$$k = 4 \qquad \textit{in the other cases.} \tag{14}$$

Proof of (12):

$$k = 1 \Leftrightarrow \mathcal{B}_3(w) = w \Leftrightarrow \mathrm{i}(\mathrm{r}(w)) = w \Leftrightarrow \mathrm{i}(\mathrm{i}(\mathrm{r}(w))) = \mathrm{i}(w) \Leftrightarrow \mathrm{r}(w) = \mathrm{i}(w).$$

Proof of (13) and (14): The key idea is to interpret the bijections i, r and s geometrically as symmetries of the permutation matrix. For $w = a_1 \ldots a_n \in \mathfrak{S}_n$, the permutation matrix $\mathcal{P}_w$ is the $n\times n$ matrix containing 1 at the intersection of the i-th row and the a_i-th column and 0 elsewhere.

For example:

$$\mathcal{P}_w = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ is the permutation matrix of } w = 13254.$$

Inverting w, that is, applying i on w, consists in transposing $\mathcal{P}_w$. Reversing w, that is, applying r on w, consists in reflecting $\mathcal{P}_w$ with respect to the central row (see Figure 1).
Composing these operations, one obtains that $\mathcal{B}_3 \equiv \mathrm{i} \circ \mathrm{r}$ consists in rotating $\mathcal{P}_w$ around the center of $\mathcal{P}_w$ (the intersection of the two axes shown in Figure 1) on the angle $-\frac{\pi}{2}$. Therefore,

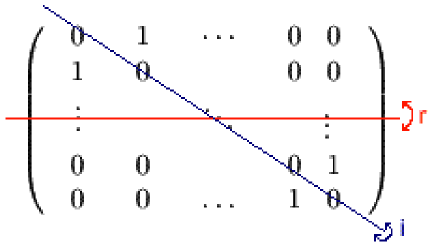

Figure 1: i in blue and r in red.

$\mathcal{B}_3^{(4)}(w) = w$, and k must divide 4. Thus k takes one of the following values: 1, 2 or 4.

For $k = 2$ one has:

$$\begin{aligned} k = 2 \quad &\Leftrightarrow \quad \mathcal{P}_w \text{ remains unchanged after rotation by angle } \pi \\ &\Leftrightarrow \quad \text{the center of } \mathcal{P}_w \text{ is a center of symmetry of } \mathcal{P}_w \\ &\Leftrightarrow \quad r(s(w)) = w \text{ as shown by figure 2.} \end{aligned}$$

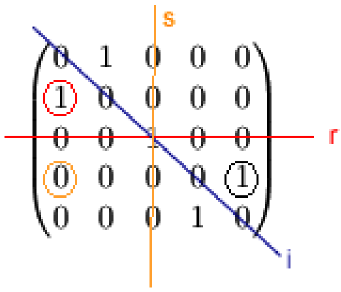

Figure 2: Applying r ∘ s to the "1" in the fourth row.

This proves theorem 4.9. ■

Note: Figure 2 gives an alternate proof of lemma 4.7: i ∘ r ∘ i ≡ s.

5 Summary

Our present consideration was dedicated to the study of general properties of good sequences using the powerful technique of bijective proofs. To start with, we have determined the total number of good sequences of fixed length by constructing a bijection $\mathcal{B}$ between a set of good sequences and a set of permutations (Theorem 2.1). We have then used bijections to find the generating function of a certain statistic on good sequences: greatest element, rank, degree, number of records, etc. For this purpose, we have constructed a new bijection $\mathcal{B}_2$ which kept the records invariant (Theorem 3.7). Using this new bijection we have studied properties of good sequences avoiding Full Sequences and we have found a relation between Eulerian polynomials and the number of ways n competitors can rank in a competition, allowing for the possibility of ties.

Furthermore, by composing these two bijections we have constructed a new bijection $\mathcal{B}_3$ from the set of permutations to itself. We have proved certain interesting facts about this bijection: $\mathcal{B}_3$ can be easily defined (theorem 3.15) and it possesses definite properties with respect to inversions, ascents, descents and its orbits.

Appendix

This appendix is devoted to a bijection described in [3]. Similarly to the double array representation, we can interpret a permutation w of length n as a bijection $[n] \to [n]$, that is, $w(i) = j \iff j$ appears in the i-th position in w. For every element $k \in [n]$ we can look at the sequence $k, w(k), w^{(2)}(k), \cdots$. Since w is a bijection, there exists a unique positive integer l such that $k, w(k), w^{(2)}(k), \cdots, w^{(l-1)}(k)$ are all different. We say that $k, w(k), w^{(2)}(k), \cdots, w^{(l-1)}(k)$ is a cycle of length l. Note that a cycle remains the same after applying a cyclic permutation on it. We can now write a permutation in its cycle form $C_1, \ldots, C_n$ where the C_i are disjoint cycles. For example, take $w = 2165473$. $w(1) = 2$, $w(2) = 1$, $w(3) = 6$, $w(4) = 5$, $w(5) = 4$, $w(6) = 7$, $w(7) = 3$. Hence $w = (12)(367)(45)$. Notice that we also have $w = (45)(12)(367) = (673)(12)(54)$. Given a permutation, we can write it in a *standard representation* by requiring that the greatest element of a cycle is placed the first in the cycle and that the subsequence formed by the first element of each cycle is increasing. For instance, the standard representation of $w = 2165473$ is $(21)(54)(736)$. Now remove the parentheses: we obtain a new permutation $\hat{w} = 2154736$. The general procedure to apply our bijection on a permutation w is the following: write w in its standard representation, erase the parentheses and get $\hat{w}$. To recover w from $\hat{w}$, write down a left parenthesis before every record since in the standard representation of w the first element of a cycle is a record, and finally complete by right parentheses when needed. Hence our bijection is a bijection $\mathfrak{S}_n \to \mathfrak{S}_n$ which maps a permutation with k cycles to a permutation with k records, and conversely.

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