Lévy processes and random discrete structures

LÉVY 2016 SUMMER SCHOOL ON LÉVY PROCESSES

PRELIMINARY VERSION

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The goal of this lecture is to study a family of random trees closely related to stable spectrally positive Lévy processes. Trees appear in many different areas such as computer science (where trees appear in the analysis of random algorithms for instance connected with data allocation), combinatorics (trees are combinatorial objects by essence), mathematical genetics (as phylogenetic trees), in statistical physics (for instance in connection with random maps as we will see below) and in probability theory (where trees describe the genealogical structure of branching processes, fragmentation processes, etc.).

In Section 1, we define Bienaymé–Galton–Watson trees and explain how they are coded by random walks. In particular, for a particular family of trees, these random walks are closely related to stable spectrally positive Lévy processes. In Section 2, we will use stable spectrally positive Lévy processes to obtain a result concerning Bienaymé–Galton–Watson trees. In Section 3, we will use Bienaymé–Galton–Watson trees to obtain a result concerning stable spectrally positive Lévy processes.

1 Bienaymé–Galton–Watson trees and their coding by random walks

1.1 Trees

In this lectures, by *tree*, we will always mean plane tree (sometimes also called rooted ordered trees). To define this notion, we follow Neveu's formalism. Let $\mathcal{U}$ be the set of labels defined by

$$\mathcal{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

where, by convention, $(\mathbb{N}^*)^0 = \{\emptyset\}$. In other words, an element of $\mathcal{U}$ is a (possible empty) sequence $u = u_1 \cdots u_j$ of positive integers. When $u = u_1 \cdots u_j$ and $v = v_1 \cdots v_k$ are elements of $\mathcal{U}$, we let $uv = u_1 \cdots u_j v_1 \cdots v_k$ be the concatenation of u and v. In particular, $u\emptyset = \emptyset u = u$. Finally, a *plane tree* is a finite subset of $\mathcal{U}$ satisfying the following three conditions:

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(i) $\emptyset \in \tau$,

(ii) if $v \in \tau$ and $v = uj$ for a certain $j \in \mathbb{N}^*$, then $u \in \tau$,

(iii) for every $u \in \tau$, there exists an integer $k_u(\tau) \geqslant 0$ such that for every $j \in \mathbb{N}^*$, $uj \in \tau$ if and only if $1 \leqslant j \leqslant k_u(\tau)$.

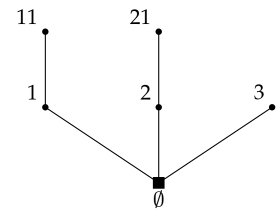

Figure 1: An example of a tree τ, where $\tau = \{\emptyset, 1, 11, 2, 21, 3\}$.

In the sequel, by *tree* we will always mean finite plane tree. We will often view the elements of τ as the individuals of a population whose τ is the genealogical tree and $\emptyset$ is the ancestor (the root). In particular, for $u \in \tau$, we say that $k_u(\tau)$ is the number of children of u, and write k_u when τ is implicit. The size of τ, denoted by $|\tau|$, is the number of vertices of τ. We denote by $\mathbb{A}$ the set of all trees and by $\mathbb{A}_n$ the set of all trees of size n. Finally, for $j \geqslant 1$, a forest of j trees is an elements of $\mathbb{A}^j$.

1.2 Bienaymé–Galton–Watson trees

We now define a probability measure on $\mathbb{A}$ which describes, roughly speaking, the law of a random tree which describes the genealogical tree of a population where individuals have a random number of children, independently, distributed according to a probability measure μ, called the offspring distribution. Such models were considered by Bienaymé [3] and Galton & Watson [11], who were interested in estimating the probability of extinction of noble names.

We will always make the following assumptions on μ:

(i) $\mu = (\mu(i) : i \geqslant 0)$ is a probability distribution on $\{0, 1, 2, \ldots\}$,

(ii) $\sum_{k \geqslant 0} k\mu(k) \leqslant 1$,

(iii) $\mu(0) + \mu(1) < 1$.

Theorem 1.1. *Set, for every* $\tau \in \mathbb{A}$,

$$\mathbb{P}_\mu(\tau) = \prod_{u \in \tau} \mu(k_u). \tag{1}$$

Then $\mathbb{P}_\mu$ *defines a probability distribution on* $\mathbb{A}$.

Before proving this result, let us mention that in principle we should define the σ-field used for $\mathbb{A}$. Here, since $\mathbb{A}$ is countable, we simply take the set of all subsets of $\mathbb{A}$ as the σ-field, and we will never mention again measurability issues (one should however be careful when working with infinite trees).

Proof of Theorem 1.1. Set $c = \sum_{\tau \in \mathbb{A}} \mathbb{P}_\mu(\tau)$. Our goal is to show that $c = 1$.

Step 1. We decompose the set of trees according to the number of children of the root and write

$$c = \sum_{k \geqslant 0} \sum_{\tau \in \mathbb{A}, k_\emptyset = k} \mathbb{P}_\mu(\tau) = \sum_{k \geqslant 0} \sum_{\tau_1 \in \mathbb{A}, \ldots, \tau_k \in \mathbb{A}} \mu(k) \mathbb{P}_\mu(\tau_1) \cdots \mathbb{P}_\mu(\tau_k) = \sum_{k \geqslant 0} \mu(k) c^k.$$

Step 2. Set, for $0 \leqslant s \leqslant 1$, $f(s) = \sum_{k \geqslant 0} \mu(k) s^k - s$. Then $f(0) = \mu(0) > 0$, $f(1) = 0$, $f'(1) = (\sum_{i \geqslant 0} i\mu(i)) - 1 < 0$ and $f'' > 0$ on $[0, 1]$. Therefore, the only solution of $f(s) = 0$ on $[0, 1]$ is $s = 1$.

Step 3. We check that $c \leqslant 1$ by constructing a random variable whose "law" is $\mathbb{P}_\mu$. To this ender, consider a collection $(K_u : u \in \mathcal{U})$ of i.i.d. random variables with same law μ (defined on the same probability space). Then set

$$\mathcal{T} \coloneqq \left\{ u_1 \cdots u_n \in \mathcal{U} : u_i \leqslant K_{u_1 u_2 \cdots u_{i-1}} \text{ for every } 1 \leqslant i \leqslant n \right\}.$$

(Intuitively, K_u represents the number of children of $u \in \mathcal{U}$ if u is indeed in the tree. Then $\mathcal{T}$ is a random plane tree, but possible infinite. But for a fixed tree $\tau \in \mathbb{T}$, we have

$$\mathbb{P}\left(\mathcal{T} = \tau\right) = \mathbb{P}\left(X_u = k_u(\tau) \text{ for every } u \in \tau\right) = \prod_{u \in \tau} \mu(k_u) = \mathbb{P}_\mu\left(\tau\right).$$

Therefore

$$c = \sum_{\tau \in \mathbb{A}} \mathbb{P}_\mu\left(\tau\right) = \sum_{\tau \in A} \mathbb{P}\left(\mathcal{T} = \tau\right) = \mathbb{P}\left(\mathcal{T} \in \mathcal{A}\right) \leqslant 1.$$

By the first two steps, we conclude that $c = 1$ and this completes the proof. □

Remark 1.2. When $\sum_{i \geqslant 0} i\mu(i) > 1$, let us mention that it is possible to define a probability measure $\mathbb{P}_\mu$ on the set of all plane (not necessarily finite) trees in such a way that the formula (1) holds for finite trees. However, since we are only interested in finite trees, we will not enter such considerations.

In the sequel, by Bienaymé–Galton–Watson tree with offspring distribution μ (or simply BGW_μ tree), we mean a random tree (that is a random variable defined on some probability space taking values in $\mathbb{A}$) whose distribution is $\mathbb{P}_\mu$. We will alternatively speak of BGW tree when the offspring distribution is implicit.

The most important tool in the study of BGW trees is their coding by random walks, which are usually well understood. The idea of coding BGW trees by functions goes back to Harris [5], and was popularized by Le Gall & Le Jan [10] et Bennies & Kersting [1]. We start by explaining the coding of deterministic trees. We refer to [9] for further applications.

1.3 Coding trees

To code a tree, we first define an order on its vertices. To this end, we use the lexicographic order $\prec$ on the set $\mathcal{U}$ of labels, for which $v \prec w$ if there exists $z \in \mathcal{U}$ with $v = z(a_1, \dots, a_n)$, $w = z(b_1, \dots, b_m)$ and $a_1 < b_1$.

If $\tau \in \mathbb{A}$, let $u_0, u_1, \dots, u_{|\tau|-1}$ be the vertices of τ ordered in lexicographic order, an recall that k_u is the number of children of a vertex u.

Definition 1.3. The *Łukasiewicz path* $\mathcal{W}(\tau) = (\mathcal{W}_n(\tau), 0 \leqslant n \leqslant |\tau|)$ of τ is defined by $\mathcal{W}_0(\tau) = 0$ and, for $0 \leqslant n \leqslant |\tau| - 1$:

$$\mathcal{W}_{n+1}(\tau) = \mathcal{W}_n(\tau) + k_{u_n}(\tau) - 1.$$

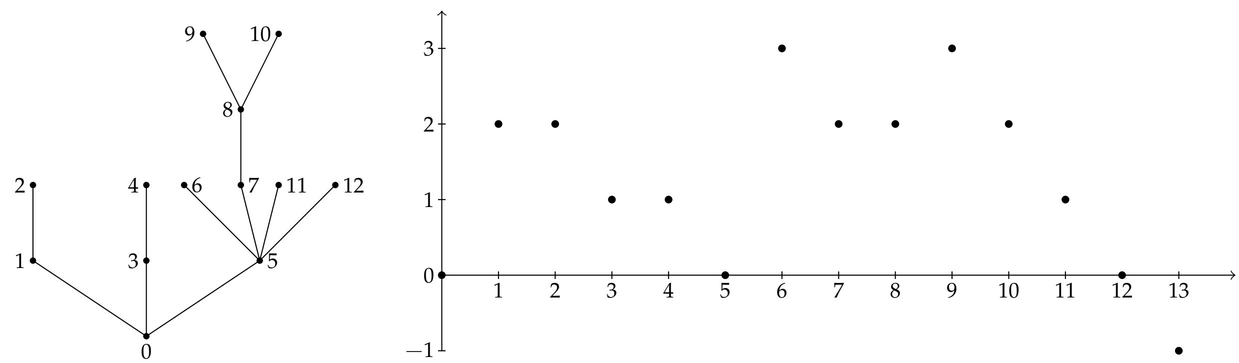

Figure 2: A tree (with its vertices numbered according to the lexicographic order) and its associated Łukasiewicz path.

See Fig. 2 for an example. Before proving that the Łukasiewicz path codes bijectively trees, we need to introduce some notation. For $n \geqslant 1$, set

$$\overline{\mathcal{S}}_n^{(1)} = \{(x_1, \dots, x_n) \in \{-1, 0, 1, \dots\} : \quad x_1 + \cdots + x_n = -1 \text{ et } x_1 + \cdots + x_j > -1 \text{ for every } 1 \leqslant j \leqslant n-1\}.$$

If $\mathbf{x} = (x_1, \dots, x_n) \in \overline{\mathcal{S}}_n^{(k)}$ and $\mathbf{y} = (y_1, \dots, y_m) \in \overline{\mathcal{S}}_m^{(k')}$, we write $\mathbf{xy} = (x_1, \dots, x_n, y_1, \dots, y_m)$ for the concatenation of $\mathbf{x}$ and $\mathbf{y}$. In particular, $\mathbf{xy} \in \overline{\mathcal{S}}_{n+m}^{(k+k')}$. If $\mathbf{x} \in \overline{\mathcal{S}}^{(k)}$, we may write $\mathbf{x} = \mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_k$ with $\mathbf{x}_i \in \overline{\mathcal{S}}^{(1)}$ for every $1 \leqslant i \leqslant k$ in a unique way.

Proposition 1.4. *For every $n \geqslant 1$, the mapping Φ_n defined by*

$$\begin{array}{rccl} \Phi_n : & \mathbb{A}_n & \longrightarrow & \overline{\mathcal{S}}_n^{(1)} \\ & \tau & \longmapsto & \left(k_{u_{i-1}} - 1 : 1 \leqslant i \leqslant n\right) \end{array}$$

is a bijection.

For $\tau \in \mathbb{A}$, set $\Phi(\tau) = \Phi_{|\tau|}(\tau)$. Proposition 1.4 shows that the Łukasiewicz indeed bijectively codes trees (because the increments of the Łukasiewicz path of τ are the elements of $\Phi(\tau)$) and that $\mathcal{W}_{|\tau|}(\tau) = -1$.

Proof. Fix $\tau \in \mathbb{A}_n$. We first check that $\Phi_n(\tau) \in \overline{\mathcal{S}}_n^{(1)}$. For every $1 \leqslant j \leqslant n$, we have

$$\sum_{i=1}^{j} \left(k_{u_{i-1}} - 1\right) = \sum_{i=1}^{j} k_{u_{i-1}} - j. \tag{2}$$

Note that the sum $\sum_{i=1}^{j} k_{u_{i-1}}$ counts the number of children of $u_0, u_1, \ldots, u_{j-1}$. If $j < n$, the vertices $u_1, \ldots, u_j$ are children of $u_0, u_1, \ldots, u_{j-1}$, so that the quantity (2) is positive. If $j = n$, the sum $\sum_{i=1}^{n} k_{u_{i-1}}$ counts vertices who have a parent, that is everyone except the root, so that this sum is $n-1$. Therefore, $\Phi_n(\tau) \in \overline{\mathcal{S}}_n^{(1)}$.

We next show that Φ_n is bijective by strong induction on n. For $n = 1$, there is nothing to do. Fix $n \geqslant 2$ and assume that Φ_j is a bijection fore very $j \in \{1, 2, \ldots, n-1\}$. Take $\mathbf{x} = (a, x_1, \ldots, x_{n-1}) \in \overline{\mathcal{S}}_n^{(1)}$. We have $\Phi_n(\tau) = \mathbf{x}$ if and only if $k_\emptyset(\tau) = a+1$, and $(x_1, \ldots, x_{n-1})$ must be the concatenation of the images by Φ of the subtrees $\tau_1, \ldots, \tau_{a+1}$ attached on the children of $\emptyset$. But $(x_1, x_1, \ldots, x_{n-1}) \in \overline{\mathcal{S}}_{n-1}^{(a+1)}$, so $(x_1, \ldots, x_{n-1}) = \mathbf{x}_1 \cdots \mathbf{x}_{a+1}$ can be written as a concatenation of elements of $\overline{\mathcal{S}}^{(1)}$ in a unique way. Hence

$$\begin{aligned} \Phi_n(\tau) = \mathbf{x} \quad &\Longleftrightarrow \quad \Phi_{|\tau_i|}(\tau_i) = \mathbf{x}_i \text{ for every } i \in \{1, 2, \ldots, a+1\} \\ &\Longleftrightarrow \quad \tau = \{\emptyset\} \cup \bigcup_{i=1}^{a+1} i\Phi_{|\tau_i|}^{-1}(\mathbf{x}_i), \end{aligned}$$

where we have used the induction hypothesis (since $|\tau_i| < |\tau|$). This completes the proof. $\square$

Extension to forests. Recall that a forest of k trees is a sequence of k trees. Proposition 1.4 is easily extended to forests by defining $\Phi(\tau_1, \ldots, \tau_k)$ as the concatenation $\Phi(\tau_1) \cdots \Phi(\tau_k)$. This yields a bijection between the set of all forests with k trees and n vertices (with $k \leqslant n$) and $\overline{\mathcal{S}}_n^{(k)}$, where

$$\begin{aligned} \overline{\mathcal{S}}_n^{(k)} \coloneqq \{(x_1, \ldots, x_n) \in \{-1, 0, 1, \ldots\} : \quad & x_1 + \cdots + x_n = -k \\ & \text{and } x_1 + \cdots + x_j > -k \text{ for every } 1 \leqslant j \leqslant n-1\}. \end{aligned}$$

Similarly, the Łukasiewicz path of a forest is defined as the concatenation of the jumps of the Łukasiewicz paths of the trees of the forest.

1.4 Coding BGW trees by random walks

We will now identify the law of the Łukasiewicz path of a BGW tree. Consider the random walk $(W_n)_{n \geqslant 0}$ on $\mathbb{Z}$ such that $W_0 = 0$ with jump distribution given by $\mathbb{P}(W_1 = k) =$

$\mu(k+1)$ for every $k \geqslant -1$. In other words, for $n \geqslant 1$, we may write

$$W_n = X_1 + \cdots + X_n,$$

where the random variables $(X_i)_{i \geqslant 1}$ are independent and identically distributed with $\mathbb{P}(X_1 = k) = \mu(k+1)$ for every $k \geqslant -1$. This random walk will play a crucial role in the sequel. Finally, for $j \geqslant 1$, set

$$\zeta_j = \inf\{n \geqslant 1 : W_n = -j\},$$

which is the first passage time of the random walk at $-j$ (which could be a priori be infinite !).

Proposition 1.5. *Let $\mathcal{T}$ be a random* BGW_μ *tree. Then the random vectors (of random length)*

$$\left(\mathcal{W}_0(\mathcal{T}), \mathcal{W}_1(\mathcal{T}), \ldots, \mathcal{W}_{|\mathcal{T}|}(\mathcal{T})\right) \qquad \text{and} \qquad (W_0, W_1, \ldots, W_{\zeta_1})$$

have the same distribution.

In particular, $|\mathcal{T}|$ and ζ_1 have the same distribution.

Proof. Fix $n \geqslant 1$ and integers $x_1, \ldots, x_n \geqslant -1$. Set

$$\begin{aligned} A &= \mathbb{P}\left(\mathcal{W}_1(\mathcal{T}) = x_1, \mathcal{W}_2(\mathcal{T}) - \mathcal{W}_1(\mathcal{T}) = x_2, \ldots, \mathcal{W}_n(\mathcal{T}) - \mathcal{W}_{n-1}(\mathcal{T}) = x_n\right), \\ B &= \mathbb{P}\left(W_1 = x_1, W_2 - W_1 = x_2, \ldots, W_n - W_{n-1} = x_n\right). \end{aligned}$$

We shall show that $A = B$.

First of all, if $(x_1, \ldots, x_n) \notin \overline{\mathcal{S}}_n^{(1)}$, then $A = B = 0$. Now, if $(x_1, \ldots, x_n) \in \overline{\mathcal{S}}_n^{(1)}$, by Proposition 1.4 there exists a tree τ whose Łukasiewicz path is $(0, x_1, x_1 + x_2, \ldots)$. Then, by (1),

$$A = \mathbb{P}(\mathcal{T} = \tau) = \prod_{u \in \tau} \mu(k_u) = \prod_{i=1}^{n} \mu(x_i + 1),$$

et

$$\begin{aligned} B &= \mathbb{P}\left(W_1 = x_1, W_2 - W_1 = x_2, \ldots, W_n - W_{n-1} = x_n, \zeta_1 = n\right) \\ &= \mathbb{P}\left(W_1 = x_1, W_2 - W_1 = x_2, \ldots, W_n - W_{n-1} = x_n\right) \\ &= \prod_{i=1}^{n} \mu(x_i + 1). \end{aligned}$$

For the second equality, we have used the equality of events $\{W_1 = x_1, W_2 - W_1 = x_2, \ldots, W_n - W_{n-1} = x_n, \zeta_1 = n\} = \{W_1 = x_1, W_2 - W_1 = x_2, \ldots, W_n - W_{n-1} = x_n\}$, which comes from the fact that $(x_1, \ldots, x_n) \in \overline{\mathcal{S}}_n^{(1)}$. Hence $A = B$, and this completes the proof. □

Extension to forests. The previous result is immediately adapted to forests:

Corollary 1.6. *For every $n \geqslant 1$, the Łukasiewicz path of a forest of n i.i.d BGW_μ trees has the same distribution as $(W_0, W_1, \dots, W_{\zeta_n})$.*

Remark 1.7. If μ is an offspring distribution with mean m, we have $\mathbb{E}[W_1] = m - 1$. Indeed,

$$\mathbb{E}[W_1] = \sum_{i \geqslant -1} i\mu(i+1) = \sum_{i \geqslant 0} (i-1)\mu(i) = m - 1.$$

In particular, $(W_n)_{n \geqslant 0}$ is a centered random walk if and only if $m = 1$ (that is if the offspring distribution is critical).

1.5 A special family of offspring distributions

The connection with Lévy processes arises when considering a special family of offspring distributions. Let us first recall some properties concerning domains of attraction.

Let $(X_i)_{i \geqslant 1}$ be i.i.d. real-valued random variables. Assume that there exists a sequence of positive real numbers $a_n \to \infty$, a sequence of real numbers (b_n) and a non-degenerate random variable U such that the convergence

$$\frac{X_1 + \cdots + X_n}{a_n} - b_n \quad \xrightarrow[n\to\infty]{(d)} \quad U$$

holds in distribution, then U is an α stable random variable for a certain value $\alpha \in (0, 2]$. We say that (the law of) X_1 is in the domain of attraction of a stable law of index α.

In the sequel, we will fix $\alpha \in (1, 2)$ and we will consider offspring distributions μ which satisfy:

(i) $\sum_{i \geqslant 0} i\mu(i) = 1$ (μ is critical),

(ii) μ is in the domain of attraction of a stable law of index α.

It is known that (ii) is equivalent to the fact that $\mu([n, \infty)) = \frac{L(n)}{n^\alpha}$ for a slowly varying function L (that is for every fixed $x > 0$, $L(ux)/L(u) \to 1$ as $u \to \infty$). A typical example to keep in mind is the case where $\mu(n) \sim \frac{c}{n^{1+\alpha}}$ for a certain constant $c > 0$.

Recalling the random walk $(W_n)_{n \geqslant 0}$ which was previously defined, W_1 is centered and also is in the domain of attraction of a stable law of index α. Therefore, there exists a sequence $a_n \to \infty$ such that

$$\frac{W_n}{a_n} \quad \xrightarrow[n\to\infty]{(d)} \quad Y^{(\alpha)}, \tag{3}$$

where $Y^{(\alpha)}$ is an α-stable spectrally positive random variable normalized so that $\mathbb{E}\left[e^{-\lambda Y^{(\alpha)}}\right] = e^{\lambda^\alpha}$ for every $\lambda \geqslant 0$. The fact that $Y^{(\alpha)}$ is spectrally positive (meaning that its Lévy measure vanishes on $\mathbb{R}_-$) follows from the fact that the negative jumps of W_1 are bounded.

Let us mention that if $\mu([n,\infty)) = \frac{L(n)}{n^\alpha}$, the sequence a_n is chosen such that

$$\frac{nL(a_n)}{n^\alpha} \quad \underset{n\to\infty}{\longrightarrow} \quad \frac{1}{|\Gamma(1-\alpha)|},$$

so that a_n is equal to $n^{1/\alpha}$ multiplied by another slowly varying function. Finally, we mention that the Lévy measure Π of $Y^{(\alpha)}$ is

$$\Pi(dr) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \cdot \frac{1}{r^{1+\alpha}} \mathbb{1}_{r>0} dr.$$

2 Maximum out-degree of Bienaymé–Galton–Watson trees

In this section, we fix $1 < \alpha < 2$ and consider a critical offspring distribution μ which belongs to the domain of attraction of a stable law of index α. For a tree $\tau \in \mathbb{A}$, we let $M(\tau) = \max_{u \in \tau} k_u(\tau)$ be the maximum number of children (or the maximum out-degree) of a vertex in τ.

The goal of this Section is to establish the following result, due to Bertoin [2], and we present the proof given in [2].

Theorem 2.1 (Bertoin)**.** *Let $\mathcal{T}$ be a* BGW_μ *tree. Then*

$$\mathbb{P}\left(M(\mathcal{T}) > x\right) \quad \underset{x\to\infty}{\sim} \quad \frac{\beta}{x},$$

where $\beta > 0$ only depends on α and is the unique positive solution of the equation

$$\sum_{n=0}^{\infty} (-1)^n \frac{\beta^n}{(n-\alpha)n!} = 0.$$

The fact that this equation has a unique solution follows for instance from the fact that $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(n-\alpha)n!}$ is continuous on $\mathbb{R}_+$, increasing with $f(0) < 0$ and $f(1) > 1$.

The first idea it to reduce the problem to a forest of BGW_μ trees.

Lemma 2.2. *For $n \geqslant 1$, denote by M_n the maximum out-degree in a forest of n i.i.d.* BGW_μ *trees. Assume that*

$$\mathbb{P}\left(M_n \leqslant n\right) \quad \underset{n\to\infty}{\longrightarrow} \quad e^{-\beta}.$$

Then

$$\mathbb{P}\left(M_1 > x\right) \quad \underset{x\to\infty}{\sim} \quad \frac{\beta}{x}.$$

Proof. We first check that

$$\mathbb{P}\left(M_1 > n\right) \quad \underset{n\to\infty}{\sim} \quad \frac{\beta}{n} \tag{4}$$

when $n \to \infty$ along integer valuers. By independence, $\mathbb{P}(M_n \leqslant n) = (1 - \mathbb{P}(M_1 > n))^n$. Hence, by assumption,

$$n \ln(1 - \mathbb{P}(M_1 > n)) \underset{n\to\infty}{\sim} -\beta.$$

Since $\mathbb{P}(M_1 > n) \to 0$ as $n \to \infty$, we have $\ln(1 - \mathbb{P}(M_1 > n)) \sim -\mathbb{P}(M_1 > n)$ as $n \to \infty$, and we readily get (4).

To finish the proof, we use a monotonicity argument by writing, for $x > 0$,

$$\mathbb{P}(M_1 > [x] + 1) > \mathbb{P}(M_1 > x) \geqslant \mathbb{P}(M_1 > [x]),$$

where $[x]$ denotes the integer part of x. By (4),

$$\mathbb{P}(M_1 > [x] + 1) \underset{x\to\infty}{\sim} \frac{\beta}{[x]+1} \underset{x\to\infty}{\sim} \frac{\beta}{x}, \qquad \mathbb{P}(M_1 > [x]) \underset{x\to\infty}{\sim} \frac{\beta}{[x]} \underset{x\to\infty}{\sim} \frac{\beta}{x}.$$

This shows that

$$\mathbb{P}(M_1 > x) \underset{x\to\infty}{\sim} \frac{\beta}{x}$$

and completes the proof. $\square$

2.1 Link with Lévy processes

A first step towards the proof of Theorem 2.1 is a connection with Lévy processes. To this end, as in Section 1.4, we introduce the random walk $(W_n)_{n\geqslant 0}$ such that $W_0 = 0$ and with jump distribution $\mathbb{P}(W_1 = i) = \mu(i+1)$ for $i \geqslant -1$. As seen was seen in Section 1.5, there exists a sequence $a_n \to \infty$ such that

$$\frac{W_n}{a_n} \xrightarrow[n\to\infty]{(d)} Y^{(\alpha)},$$

where $Y^{(\alpha)}$ is an α-stable spectrally positive random variable normalized so that $\mathbb{E}\left[e^{-\lambda Y^{(\alpha)}}\right] = e^{\lambda^\alpha}$ for every $\lambda \geqslant 0$.

We will need a functional extension of this result (known as Donsker's invariance principle in the case of brownian motion). Before, let us recall several facts concerning the Skorokhod topology. Denote by $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ the space of all real-valued càdlàg functions defined on $\mathbb{R}_+$, equipped with the Skorokhod J_1 metric so that $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ is a Polish metric space (meaning that it is complete and separable). Recall (see e.g. [6, Theorem 1.14 in Chapter VI]) that if $f_n, f \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ the convergence $f_n \to f$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ holds if and only if there exist time changes λ_n such that:

– $\lambda_n(0) = 0$, λ_n is strictly increasing and continuous, $\lambda_n(\infty) = \infty$,

– $\sup_{t\geqslant 0} |\lambda_n(t) - t| \xrightarrow[n\to\infty]{} 0$,

– for every integer $N \geqslant 1$, $\sup_{t \leqslant N} |f_n(\lambda_n(t)) - f(t)| \quad \underset{n\to\infty}{\longrightarrow} \quad 0.$

Then the convergence

$$\left(\frac{W_{[nt]}}{a_n} : t \geqslant 0\right) \quad \xrightarrow[n\to\infty]{(d)} \quad Y$$

holds in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ as $n \to \infty$ (see Theorem 16.14 in [7]). Alternatively, there exists a sequence $\widetilde{a}_n \to \infty$ such that the convergence

$$\left(\frac{W_{[\widetilde{a}_n t]}}{n} : t \geqslant 0\right) \quad \xrightarrow[n\to\infty]{(d)} \quad Y \tag{5}$$

holds in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ as $n \to \infty$.

We now introduce some notation. Let $(Y_t)_{t\geqslant 0}$ be an α-stable spectrally positive Lévy process with $Y_0 = 0$ normalised so that $\mathbb{E}\left[e^{-\lambda Y_1}\right] = e^{\lambda^\alpha}$ for every $\lambda \geqslant 0$. For $s, t > 0$ we set

$$\Delta Y_s = Y_s - Y_{s-}, \qquad \Delta^*_t = \sup_{s<t} \Delta Y_s,$$

with the convention $\Delta Y_0 = 0$ and $\Delta^*_0 = 0$. Finally, set

$$\tau_1 = \inf\{t \geqslant 0 : Y_t < -1\}.$$

We are now ready to state and prove the connection with Lévy processes.

Proposition 2.3. *Then the convergence*

$$\frac{M_n}{n} \quad \xrightarrow[n\to\infty]{(d)} \quad \Delta^*_{\tau_1}$$

holds in distribution.

Proof. Since $\Delta Y_{\tau_1} = 0$ almost surely, by continuity properties of the J_1 metric (see [6, Proposition 2.12 in Chapter VI]), the convergence (8) implies the convergence in distribution of the càdlàg functions stopped at the first time they hit -1. In other words,

$$\left(\frac{W_{[\widetilde{a}_n t]}}{n} : 0 \leqslant t \leqslant \frac{\zeta_n}{\widetilde{a}_n}\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (Y_t : 0 \leqslant t \leqslant \tau_1), \tag{6}$$

where we recall that $\zeta_n = \inf\{k \geqslant 0 : W_k = -n\}$.

By Corollary 1.6, $(W_0, \dots, W_{\zeta_n})$ has the same distribution as the Łukasiewicz path of a forest of n i.i.d. BGW_μ trees. But by construction of the Łukasiewicz path, the maximum outdegree of the forest, is the maximum jump of its Łukasiewicz path plus one. Therefore, $\frac{M_n - 1}{n}$ has the same distribution as the maximum jump of $\left(\frac{W_{[\widetilde{a}_n t]}}{n} : 0 \leqslant t \leqslant \frac{\zeta_n}{\widetilde{a}_n}\right)$. The desired result follows by the continuity of the largest jump of càdlàg functions for the J_1 metric on compact time sets (see [6, Proposition 2.4 in Chapter VI]). □

The goal is now to show that $\mathbb{P}\left(\Delta^*_{\tau_1} \leqslant x\right) = e^{-\frac{\beta}{x}}$ for every $x > 0$.

2.2 A fluctuation identity for spectrally positive Lévy processes

Here we calculate the Laplace transform of hitting times for general spectrally positive Lévy processes, a result which will be needed in the calculation of $\mathbb{P}\left(\Delta^*_{\tau_1} \leqslant x\right)$.

Let $(\widetilde{Y}_t, t \geqslant 0)$ be a Lévy process with Lévy measure Π. It is known (see e.g. [8, Theorem 3.6]) that for every $\lambda \in \mathbb{R}$,

$$\forall t \ \geqslant 0,\ \mathbb{E}\left[e^{\lambda \widetilde{Y}_t}\right] < \infty \quad \Longleftrightarrow \quad \int_{|x|>1} e^{\lambda x}\,\Pi(dx) < \infty$$

Now assume that $\widetilde{Y}$ is spectrally positive, that is $\Pi = 0$ on $\mathbb{R}_-$ and that $\Pi \neq 0$ on $\mathbb{R}_+$. As a consequence, $\mathbb{E}\left[e^{-\lambda \widetilde{Y}_t}\right] < \infty$ for every $t, \lambda \geqslant 0$, and we may define the Laplace exponent Ψ by

$$\mathbb{E}\left[e^{-\lambda \widetilde{Y}_t}\right] = e^{t\Psi(\lambda)}$$

for $\lambda, t \geqslant 0$. In particular,

$$\mathbb{E}\left[e^{-\lambda \widetilde{Y}_t - t\Psi(\lambda)}\right] = 1. \tag{7}$$

The Laplace exponent Ψ is continuous on $\mathbb{R}_+$, is strictly convex and satisfies $\Psi(0) = 0$, $\Psi(\infty) = \infty$ and $\Psi'(0) = -\mathbb{E}\left[\widetilde{Y}_1\right]$ (see [8, Exercise 3.5]).

Finally, for every $q \geqslant 0$ set

$$\Phi(q) = \sup\{\lambda \geqslant 0 : \Psi(\lambda) = q\}$$

and for every $x > 0$ set

$$\tilde{\tau}_x = \inf\left\{t > 0 : \widetilde{Y}_t < -x\right\}.$$

Note that for $q \geqslant 0$, the equation $\Psi(\lambda) = q$ has only solution, except when $q = 0$ and $\mathbb{E}\left[\widetilde{Y}_1\right] > 0$.

Theorem 2.4. *For every* $q \geqslant 0$,

$$\mathbb{E}\left[e^{-q\tilde{\tau}_x} \mathbb{1}_{\tilde{\tau}_x < \infty}\right] = e^{-x\Phi(q)}.$$

In particular, $\mathbb{P}\left(\tilde{\tau}_x < \infty\right) = 1$ *if and only if* $\mathbb{E}\left[\widetilde{Y}_1\right] \leqslant 0$.

Proof. We follow the proof of [8, Theorem 3.12]. Denote by $\mathcal{F}_t$ the $\mathbb{P}$-completed σ-field generated by $\sigma(\widetilde{Y}_s, s \leqslant t)$.

We start with the case $q > 0$. For $t > 0$, using the fact that $\widetilde{Y}_{\tilde{\tau}_x} = -x$ on the event $\{\tilde{\tau}_x < \infty\}$ since $\widetilde{Y}$ is spectrally positive, write

$$\begin{aligned}
\mathbb{E}\left[e^{-\Phi(q)\widetilde{Y}_t - qt} \,\middle|\, \mathcal{F}_{\tilde{\tau}_x}\right] &= e^{-\Phi(q)\widetilde{Y}_t - qt}\mathbb{1}_{t \leqslant \tilde{\tau}_x} + \mathbb{E}\left[e^{-\Phi(q)(\widetilde{Y}_t - \widetilde{Y}_{\tilde{\tau}_x} + \widetilde{Y}_{\tilde{\tau}_x}) - q(t - \tilde{\tau}_x + \tilde{\tau}_x)}\mathbb{1}_{t > \tilde{\tau}_x} \,\middle|\, \mathcal{F}_{\tilde{\tau}_x}\right] \\
&= e^{-\Phi(q)\widetilde{Y}_t - qt}\mathbb{1}_{t \leqslant \tilde{\tau}_x} + e^{x\Phi(q) - q\tilde{\tau}_x}\mathbb{E}\left[e^{-\Phi(q)(\widetilde{Y}_t - \widetilde{Y}_{\tilde{\tau}_x}) - q(t - \tilde{\tau}_x)}\mathbb{1}_{t > \tilde{\tau}_x} \,\middle|\, \mathcal{F}_{\tilde{\tau}_x}\right] \\
&= e^{-\Phi(q)\widetilde{Y}_t - qt}\mathbb{1}_{t \leqslant \tilde{\tau}_x} + e^{x\Phi(q) - q\tilde{\tau}_x}\mathbb{1}_{t > \tilde{\tau}_x} \\
&= e^{-\Phi(q)\widetilde{Y}_{t \wedge \tilde{\tau}_x} - qt \wedge \tilde{\tau}_x}
\end{aligned}$$

where we have combined the strong Markov property with (7) for the penultimate equality. Taking expectations and using (7) again, we get that:

$$\mathbb{E}\left[e^{-\Phi(q)\widetilde{Y}_{t\wedge\tilde{\tau}_x}-qt\wedge\tilde{\tau}_x}\right]=1.$$

Then take $t\to\infty$, and since $e^{-\Phi(q)\widetilde{Y}_{t\wedge\tilde{\tau}_x}-qt\wedge\tilde{\tau}_x}\leqslant e^{x\Phi(q)}$, we get the desired result by dominated convergence.

The case $q=0$ is settled by taking $q\downarrow 0$ in the identity which has just been established. □

2.3 End of the proof

Theorem 2.1 will readily follow from Lemma 2.2 and Proposition 2.3 once the following result is established.

Proposition 2.5. *For every $x>0$, we have*

$$\mathbb{P}\left(\Delta^*_{\tau_1}\leqslant x\right)=\frac{\beta}{x}.$$

Proof. Denote by $\Pi(dr)=\frac{c}{r^{1+\alpha}}\mathbb{1}_{r>0}dr$ the Lévy measure of Y (we will not need the explicit value of the constant c).

The idea is to consider the process Y obtained by suppressing jumps greater than or equal to x by writing

$$Y_t=\underbrace{\left(Y_t-\sum_{s\leqslant t}\Delta_s\mathbb{1}_{\Delta_s>x}\right)}_{\widetilde{Y}_t}+\underbrace{\sum_{s\leqslant t}\Delta_s\mathbb{1}_{\Delta_s>x}}_{Z_t}.$$

By the Lévy-Itô decomposition, the processes $\widetilde{Y}$ and Z are independent and Z is a compound poisson process.

We start with calculating the Laplace exponent $\widetilde{\Psi}$ of $\widetilde{Y}$. First note that for $\lambda\geqslant 0$,

$$\mathbb{E}\left[e^{-\lambda Y_1}\right]=\exp\left(\lambda^{\alpha}\right)=\exp\left(\int_0^{\infty}\left(e^{-\lambda u}-1+\lambda u\right)\Pi(du)\right)$$

and

$$\mathbb{E}\left[e^{-\lambda Z_1}\right]=\exp\left(\int_x^{\infty}\left(e^{-\lambda u}-1\right)\Pi(du)\right).$$

Since $\mathbb{E}\left[e^{-Y_1}\right]=\mathbb{E}\left[e^{-\widetilde{Y}_1}e^{-Z_1}\right]=\mathbb{E}\left[e^{-\widetilde{Y}_1}\right]\mathbb{E}\left[e^{-Z_1}\right]$ by independence, we get that

$$\begin{aligned}\widetilde{\Psi}(\lambda) &= \int_0^{\infty}\left(e^{-\lambda u}-1+\lambda u\right)\Pi(du)-\int_x^{\infty}\left(e^{-\lambda u}-1\right)\Pi(du)\\ &= c\int_0^{x}\left(e^{-\lambda u}-1+\lambda u\right)\frac{1}{u^{1+\alpha}}\,du-c\lambda\int_x^{\infty}\frac{u}{u^{1+\alpha}}\,du\\ &= c\int_0^{x}\left(e^{-\lambda u}-1+\lambda u\right)\frac{1}{u^{1+\alpha}}\,du-\frac{c\lambda}{(\alpha-1)u^{\alpha-1}}\end{aligned}$$

Now, to finish the proof, introduce

$$\tilde{\tau}_1 = \inf\left\{t > 0 : \widetilde{Y}_t < -1\right\}, \qquad J = \inf\{t > 0 : \Delta Y_t > x\} = \inf\{t > 0 : \Delta Z_t > x\}$$

and note that there is the equality of events

$$\left\{\Delta^*_{\tau_1} \leqslant x\right\} = \{\tilde{\tau}_1 < J\}.$$

But $\tilde{\tau}_1$ is a measurable function of $\widetilde{Y}$ and J is a measurable function of Z, so $\tilde{\tau}_1$ and J are independent. In addition, J is distributed according to an exponential random variable of parameter

$$\Pi([x, \infty)) = \int_x^\infty \Pi(du) = \frac{c}{\alpha x^\alpha}.$$

Therefore

$$\mathbb{P}\left(\Delta^*_{\tau_1} \leqslant x\right) = \mathbb{P}\left(\tilde{\tau}_1 < J\right) = \mathbb{E}\left[e^{-\frac{c}{\alpha x^\alpha}\tilde{\tau}_1}\right].$$

Since $\mathbb{E}\left[\widetilde{Y}_1\right] \leqslant \mathbb{E}\left[Y_1\right] = 0$, by Theorem 2.4,

$$\mathbb{E}\left[e^{-\frac{c}{\alpha x^\alpha}\tilde{\tau}_1}\right] = e^{-p(x)}$$

where $p(x)$ is the positive solution of

$$c\int_0^x \left(e^{-p(x)u} - 1 + p(x)u\right) \frac{1}{u^{1+\alpha}}\, du - \frac{cp(x)}{(\alpha-1)u^{\alpha-1}} = \frac{c}{\alpha x^\alpha}.$$

The change of variable $u = xv$ in the integral gives

$$\int_0^1 \left(e^{-xp(x)v} - 1 + xp(x)v\right) \frac{1}{u^{1+\alpha}}\, du - \frac{xp(x)}{\alpha - 1} = \frac{1}{\alpha}.$$

Setting $\beta = xp(x)$, we see that β satisfies the question

$$\int_0^1 \left(e^{-\beta v} - 1 + \beta v\right) \frac{1}{u^{1+\alpha}}\, du - \frac{\beta}{\alpha - 1} = \frac{1}{\alpha}.$$

Expanding $e^{-\beta v}$ then readily gives the desired result. $\square$

3 An identity for stable spectrally positive Lévy processes

3.1 The cyclic lemma

For $1 \leqslant k \leqslant n$, set

$$\mathcal{S}_n^{(k)} := \{(x_1, \ldots, x_n) \in \{-1, 0, 1, \ldots\} : x_1 + \cdots + x_n = -k\},$$

and

$$\overline{\mathcal{S}}_n^{(k)} = \{(x_1, \ldots, x_n) \in \mathcal{S}_n^{(k)} : x_1 + \cdots + x_j > -k \text{ pour tout } 1 \leqslant j \leqslant n-1\}.$$

In the following, we identify an element of $\mathbb{Z}/n\mathbb{Z}$ with its unique representative in $\{0, 1, \ldots, n-1\}$. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{S}_n^{(k)}$ and $i \in \mathbb{Z}/n\mathbb{Z}$, we set

$$\mathbf{x}^{(i)} = (x_{i+1}, \ldots, x_{i+n}),$$

where the addition of indices is considered modulo n. We say that $\mathbf{x}^{(i)}$ is obtained from $\mathbf{x}$ by a cyclic permutation. Note that $\mathcal{S}_n^{(k)}$ is stable by cyclic permutations.

Definition 3.1. For $\mathbf{x} \in \mathcal{S}_n^{(k)}$, set

$$I_{\mathbf{x}} = \left\{ i \in \mathbb{Z}/n\mathbb{Z} : \mathbf{x}^{(i)} \in \overline{\mathcal{S}}_n^{(k)} \right\}.$$

See Fig. 3 for an example.

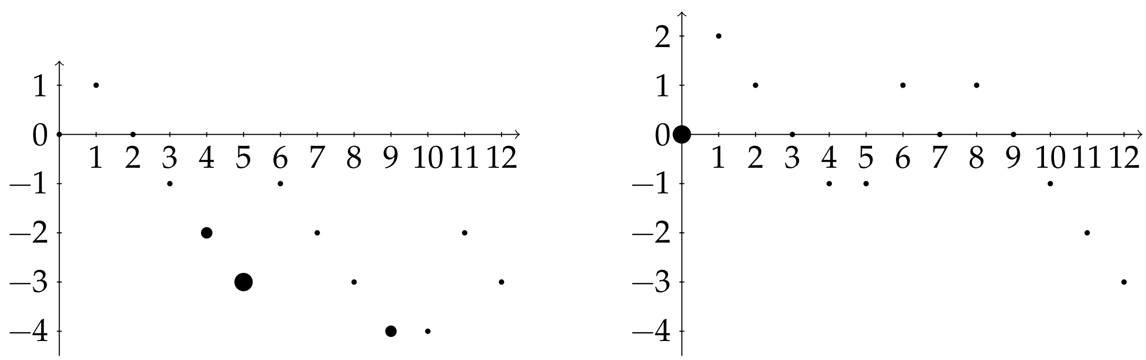

Figure 3: For $\mathbf{x} = (x_1, x_2, \ldots)$, we represent $x_1 + \cdots + x_i$ as a function of i. On the left, we take $\mathbf{x} = (1, -1, -1, -1, -1, 2, -1, -1, -1, 0, 2, -1) \in S_{13}^{(3)}$, where $I_{\mathbf{x}} = \{4, 5, 9\}$. On the right, we take $\mathbf{x}^{(5)}$, which is indeed an element of $\overline{S}_{13}^{(3)}$.

Note that if $\mathbf{x} \in \mathcal{S}_n^{(k)}$ and $i \in \mathbb{Z}/n\mathbb{Z}$, then $\mathrm{Card}(I_{\mathbf{x}}) = \mathrm{Card}(I_{\mathbf{x}^{(i)}})$.

Theorem 3.2. *(Cyclic Lemma) For every $\mathbf{x} \in \mathcal{S}_n^{(k)}$, we have* $\mathrm{Card}(I_{\mathbf{x}}) = k$.

Therefore, if $\mathbf{x} \in \cup_{n \geqslant k} \mathcal{S}_n^{(k)}$, the set $I_{\mathbf{x}}$ depends on $\mathbf{x}$, but its cardinal does not depend on $\mathbf{x}$!

Proof. We start with an intermediate result: we check that $\mathrm{Card}(I_{\mathbf{x}})$ does not change if one concatenates

$$(a, \underbrace{-1, \ldots, -1}_{a \text{ times}})$$

to the left of $\mathbf{x}$, for an integer $a \geqslant 1$. To this end, fix $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{S}_n^{(k)}$ and set

$$\widetilde{\mathbf{x}} = (a, \underbrace{-1, \ldots, -1}_{a \text{ times}}, x_1, \ldots, x_n).$$

First, it is clear that $0 \in I_{\widetilde{\mathbf{x}}}$ if and only if $0 \in I_{\mathbf{x}}$. Then, if $0 < j \leqslant n-1$, we have

$$\widetilde{\mathbf{x}}^{(j+a+1)} = (x_{j+1}, \dots, x_n, a, -1, \dots, -1, x_1, \dots, x_j).$$

It readily follows that $j \in I_{\mathbf{x}}$ if and only if $j + a + 1 \in I_{\widetilde{\mathbf{x}}}$. Next, we check that if $0 < i \leqslant a+1$, then $i \notin I_{\widetilde{\mathbf{x}}}$. Indeed, if $0 < i \leqslant a+1$, then

$$\widetilde{\mathbf{x}}^{(i)} = (\underbrace{-1, \dots, -1}_{a-i+1 \text{ times}}, x_1, x_2, \dots, x_n, a, -1, \dots, -1).$$

The sum of the elements of $\widetilde{\mathbf{x}}^{(i)}$ up to element x_n is

$$x_1 + \cdots + x_n - (a - i + 1) = -k - (a - i + 1) \leqslant -k.$$

Hence $\widetilde{\mathbf{x}}^{(i)} \notin I_{\widetilde{\mathbf{x}}}$. This shows our intermediate result.

Let us now establish the Cyclic Lemma by strong induction on n. For $n = k$, there is nothing to do, as the only element of $\mathcal{S}_n^{(k)}$ is $\mathbf{x} = (-1, -1, \dots, -1)$. Then consider an integer $n > k$ such that the Cyclic Lemma holds for elements of $\mathcal{S}_j^{(k)}$ with $j = k, \dots, n-1$. Take $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{S}_n^{(k)}$. Since $\mathrm{Card}(I_{\mathbf{x}})$ does not change under cyclic permutations of $\mathbf{x}$ and since there exists $i \in \{1, 2, \dots, n\}$ such that $x_i \geqslant 0$ (because $n > k$), without loss of generality we may assume that $x_1 \geqslant 0$. Denote by $1 = i_1 < i_2 < \cdots < i_m$ the indices i such that $x_i \geqslant 0$ and set $i_{m+1} = n+1$ by convention. Then

$$-k = \sum_{i=1}^{n} x_i = \sum_{j=1}^{m} \Big(x_{i_j} - (i_{j+1} - i_j - 1) \Big)$$

since $i_{j+1} - i_j - 1$ count the number of consecutive -1 that immediately follows x_{i_j}. Since this sum is negative, there exists $j \in \{1, 2, \dots, m\}$ such that $x_{i_j} \leqslant i_{j+1} - i_j - 1$. Thefore x_{i_j} is immediately followed by at least x_{i_j} consective times -1. Then let $\widetilde{\mathbf{x}}$ be the vector obtained from $\mathbf{x}$ by supressing x_{i_j} immediately followed by x_{i_j} times -1, so that $\mathrm{Card}(I_{\widetilde{\mathbf{x}}}) = \mathrm{Card}(I_{\mathbf{x}})$ by the intermediate result. Hence $\mathrm{Card}(I_{\mathbf{x}}) = k$ by induction hypothesis. □

Remark 3.3. Fix $\mathbf{x} = (x_1, \dots, x_n) \in S_n^{(k)}$. Set $m = \min\{x_1 + \cdots + x_i : 1 \leqslant i \leqslant n\}$, and $\zeta_i(\mathbf{x}) = \min\{j \geqslant 1 : x_1 + \cdots + x_j = m + i - 1\}$ for $1 \leqslant i \leqslant k$. Then

$$I_{\mathbf{x}} = \{\zeta_1(\mathbf{x}), \dots, \zeta_k(\mathbf{x})\}.$$

Indeed, this follows from the fact that this property is invariant under insertion of $(a, -1, \dots, -1)$ for an integer $a \geqslant 1$ (where -1 is written a times).

3.2 Applications to random walks

In this section, we fix a random walk $(W_n = X_1 + \cdots + X_n)_{n \geqslant 0}$ on $\mathbb{Z}$ such that $W_0 = 0$, $\mathbb{P}(W_1 \geqslant -1) = 1$ and $\mathbb{P}(W_1 = 0) < 1$. We set, for $k \geqslant 1$,

$$\zeta_k = \inf\{i \geqslant 0 : W_i = -k\}.$$

Definition 3.4. A function $F : \mathbb{Z}^n \to \mathbb{R}$ is said to be invariant under cyclic permutations if

$$\forall\, \mathbf{x} \in \mathbb{Z}^n, \quad \forall\, i \in \mathbb{Z}/n\mathbb{Z}, \quad F(\mathbf{x}) = F(\mathbf{x}^{(i)}).$$

Let us give several example of functions invariant by cyclic permutations. If $\mathbf{x} = (x_1, \dots, x_n)$, one may take $F(\mathbf{x}) = \max(x_1, \dots, x_n)$, $F(\mathbf{x}) = \min(x_1, \dots, x_n)$, $F(\mathbf{x}) = x_1 x_2 \cdots x_n$, $F(\mathbf{x}) = x_1 + \cdots + x_n$, or more generally $F(\mathbf{x}) = x_1^\lambda + \cdots + x_n^\lambda$ avec $\lambda > 0$. If $A \subset \mathbb{Z}$,

$$F(\mathbf{x}) = \sum_{i=1}^{n} \mathbb{1}_{x_i \in A},$$

which counts the number of elements in A, is also invariant under cyclic permutations. If F is invariant under cyclic permutations and $g : \mathbb{R} \to \mathbb{R}$ is a function, then $g \circ F$ is also invariant under cyclic permutations. Finally, $F(x_1, x_2, x_3) = (x_2 - x_1)^3 + (x_3 - x_2)^3 + (x_1 - x_3)^3$ is invariant under cyclic permutations but not invariant under all permutations.

Proposition 3.5. *Let $F : \mathbb{Z}^n \to \mathbb{R}$ be a function invariant under cyclic permutations. Then for every integers $k \leqslant n$ the following assertions hold.*

(i) $\mathbb{E}\left[F(X_1, \dots, X_n)\mathbb{1}_{\zeta_k = n}\right] = \frac{k}{n}\mathbb{E}\left[F(X_1, \dots, X_n)\mathbb{1}_{W_n = -k}\right]$,

(ii) $\mathbb{P}\left(\zeta_k = n\right) = \frac{k}{n}\mathbb{P}\left(W_n = -k\right)$.

The assertion (ii) is known as Kemperman's formula.

Proof. The second assertion follows from the first one simply by taking $F \equiv 1$. For (i), to simplify notation, set $\mathbf{X}_n = (X_1, \dots, X_n)$. Note that the following equalities of events hold

$$\{W_n = -k\} = \left\{\mathbf{X}_n \in \mathcal{S}_n^{(k)}\right\} \qquad \text{and} \qquad \{\zeta_k = n\} = \left\{\mathbf{X}_n \in \overline{\mathcal{S}}_n^{(k)}\right\}.$$

In particular,

$$\mathbb{E}\left[F(X_1, \dots, X_n)\mathbb{1}_{\zeta_k = n}\right] = \mathbb{E}\left[F(\mathbf{X}_n)\mathbb{1}_{\mathbf{X}_n \in \overline{\mathcal{S}}_n^{(k)}}\right].$$

Then write

$$\begin{aligned}
\mathbb{E}\left[F(\mathbf{X}_n)\mathbb{1}_{\mathbf{X}_n \in \overline{\mathcal{S}}_n^{(k)}}\right] &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[F(\mathbf{X}_n^{(i)})\mathbb{1}_{\mathbf{X}_n^{(i)} \in \overline{\mathcal{S}}_n^{(k)}}\right] \quad \text{(since } \mathbf{X}_n^{(i)} \text{ and } \mathbf{X}_n \text{ have the same law)}\\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[F(\mathbf{X}_n)\mathbb{1}_{\mathbf{X}_n^{(i)} \in \overline{\mathcal{S}}_n^{(k)}}\right] \quad \text{(invariance of } F \text{ by cyclic permutations)}\\
&= \frac{1}{n}\mathbb{E}\left[F(\mathbf{X}_n)\mathbb{1}_{\mathbf{X}_n \in \mathcal{S}_n^{(k)}}\left(\sum_{i=1}^{n} \mathbb{1}_{\mathbf{X}_n^{(i)} \in \overline{\mathcal{S}}_n^{(k)}}\right)\right]\\
&= \frac{k}{n}\mathbb{E}\left[F(X_1, \dots, X_n)\mathbb{1}_{\mathbf{X}_n \in \mathcal{S}_n^{(k)}}\right],
\end{aligned}$$

where the last equality is a consequence of the equality of the random variables

$$\mathbb{1}_{\mathbf{X}_n \in \mathcal{S}_n^{(k)}} \left(\sum_{i=1}^{n} \mathbb{1}_{\mathbf{X}_n^{(i)} \in \overline{\mathcal{S}}_n^{(k)}} \right) = k \mathbb{1}_{\mathbf{X}_n \in \mathcal{S}_n^{(k)}}$$

by the Cyclic Lemma. We conclude that

$$\mathbb{E}\left[F(\mathbf{X}_n) \mathbb{1}_{\mathbf{X}_n \in \overline{\mathcal{S}}_n^{(k)}}\right] = \frac{k}{n} \mathbb{E}\left[F(X_1, \ldots, X_n) \mathbb{1}_{W_n = -k}\right],$$

which is the desired result. □

We now present two applications of this result.

Proposition 3.6. *Assume that* $\mathbb{E}[W_1] \leqslant 0$*. Then, for every* $k \geqslant 1$*,*

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}(W_n = -k) = \frac{1}{k}.$$

Proof. We first check that $\mathbb{P}(\zeta_n < \infty) = 1$ for every $n \geqslant 1$. By the strong Markov property, it is enough to show that $\mathbb{P}(\zeta_1 < \infty) = 1$.

To this end, define the offspring distribution μ by $\mu(i) = \mathbb{P}(W_1 = i - 1)$ for $i \geqslant 0$ and let $\mathcal{T}$ be a BGW_μ random tree. By Corollary 1.6, for $n \geqslant 1$, we have $\mathbb{P}(\zeta_1 = n) = \mathbb{P}(|\mathcal{T}| = n)$. Since $\mathbb{E}[W_1] \leqslant 0$, we have $\sum_{i \geqslant 0} i\mu(i) \leqslant 1$, so that by Theorem 1.1 we have

$$1 = \sum_{n \geqslant 1} \mathbb{P}(|\mathcal{T}| = n) = \sum_{n \geqslant 1} \mathbb{P}(\zeta_1 = n) = \mathbb{P}(\zeta_1 < \infty).$$

Next, by Proposition 3.5 (ii),

$$\sum_{n=1}^{\infty} \frac{k}{n} \mathbb{P}(W_n = -k) = \sum_{n=1}^{\infty} \mathbb{P}(\zeta_k = n) = \mathbb{P}(\zeta_k < \infty) = 1,$$

and this completes the proof. □

Proposition 3.6 gives the following interesting deterministic identity:

Corollary 3.7. *For every* $0 \leqslant \lambda \leqslant 1$*,*

$$\sum_{n \geqslant 1} \frac{(\lambda n)^{n-1} e^{-\lambda n}}{n} = 1.$$

Proof. In Proposition 3.6, we take $k = 1$ and $W_1 = \mathsf{Poisson}(\lambda) - 1$. In particular, $W_n + n$ is distributed according to a Poisson random variable of parameter λn. Therefore, by Proposition 3.6,

$$1 = \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}(W_n = -1) = \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}(W_n + n = n - 1) = \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{(\lambda n)^{n-1} e^{-\lambda n}}{(n-1)!},$$

and the desired result immediately follows. □

Proposition 3.8 (Chen, Curien & Maillard [4]). *Assume that* $\mathbb{E}[W_1] \leqslant 0$. *Let* $f : \mathbb{Z} \to \mathbb{R}_+$ *be a function. Then, for every* $k \geqslant 2$,

$$\mathbb{E}\left[\frac{1}{\zeta_k - 1}\sum_{i=1}^{\zeta_k} f(X_i)\right] = \mathbb{E}\left[f(X_1)\frac{k}{k+X_1}\right].$$

Proof. Since $k \geqslant 2$, we have $\zeta_k \geqslant 2$. Now, for $n \geqslant 2$, we have by Proposition 3.5 (i):

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{n} f(X_i)\mathbb{1}_{\zeta_k=n}\right] &= \frac{k}{n}\,\mathbb{E}\left[\sum_{i=1}^{n} f(X_i)\mathbb{1}_{W_n=-k}\right] \\
&= k\,\mathbb{E}\left[f(X_1)\mathbb{1}_{W_n=-k}\right] \\
&= k\,\mathbb{E}\left[\mathbb{E}\left[f(X_1)\mathbb{1}_{W_n=-k} \,\middle|\, X_1\right]\right] \\
&= k\mathbb{E}\left[f(X_1)\mathbb{E}\left[\mathbb{1}_{W_n=-k} \,\middle|\, X_1\right]\right] \\
&= k\mathbb{E}\left[f(X_1)\mathbb{P}\left(W_{n-1} = -k - X_1 | X_1\right)\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{\zeta_k - 1}\sum_{i=1}^{\zeta_k} f(X_i)\right] &= \sum_{n\geqslant 2}\frac{1}{n-1}\mathbb{E}\left[\sum_{i=1}^{n} f(X_i)\mathbb{1}_{\zeta_k=n}\right] \\
&= \mathbb{E}\left[kf(X_1)\sum_{n\geqslant 2}\frac{1}{n-1}\mathbb{P}\left(W_{n-1} = -k - X_1\right)\right] \\
&= \mathbb{E}\left[f(X_1)\frac{k}{k+X_1}\right]
\end{aligned}$$

where we have used Proposition 3.6 for the last equality. This completes the proof. □

3.3 Application to stable spectrally positive Lévy processes

We will now pass the identity of Proposition 3.8 through the scaling limit to obtain an identity concerning stable spectrally positive Lévy processes.

As in Section 2.1, we let $(Y_t)_{t\geqslant 0}$ be an α-stable spectrally positive Lévy process with $Y_0 = 0$ normalised so that $\mathbb{E}\left[e^{-\lambda Y_1}\right] = e^{\lambda^\alpha}$ for every $\lambda \geqslant 0$. For $s > 0$ we set $\Delta Y_s = Y_s - Y_{s-}$ wih the convention $\Delta Y_0 = 0$. Finally, set

$$\tau_1 = \inf\{t \geqslant 0 : Y_t < -1\}.$$

Recall that the Lévy measure of Y is

$$\Pi(dr) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\cdot\frac{1}{r^{1+\alpha}}\mathbb{1}_{r>0}\,dr.$$

Proposition 3.9 (Chen, Curien & Maillard [4]). *For every measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$, we have*

$$\mathbb{E}\left[\frac{1}{\tau_1}\sum_{s\leqslant\tau_1} f(\Delta_s)\right] = \int_0^\infty \frac{f(x)}{1+x}\,\Pi(dx).$$

Proof. By linearity and standard approximation techniques, it is enough to establish the identity with f of the form $f(x) = \mathbb{1}_{x\geqslant a}$ with fixed $a > 0$.

Choose $(W_n = X_1 + \cdots + X_n)$ such that $W_0 = 0$, $\mathbb{P}(W_1 \geqslant -1) = 1$, $\mathbb{E}[W_1 = 0]$ and

$$\mathbb{P}(W_1 \geqslant n) \underset{n\to\infty}{\sim} \frac{c}{n^{1+\alpha}}$$

with $c = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}$. Then, as in Section 2.1, the convergence

$$\left(\frac{W_{[n^\alpha t]}}{n} : t \geqslant 0\right) \xrightarrow[n\to\infty]{(d)} Y \tag{8}$$

holds in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ as $n \to \infty$. In addition, by (6), we have

$$\left(\frac{W_{[n^\alpha t]}}{n} : 0 \leqslant t \leqslant \frac{\zeta_n}{n^\alpha}\right) \xrightarrow[n\to\infty]{(d)} (Y_t : 0 \leqslant t \leqslant \tau_1).$$

Therefore

$$\frac{1}{\frac{\zeta_n-1}{n^\alpha}}\sum_{i=1}^{\zeta_n} f\left(\frac{X_i}{n}\right) \xrightarrow[n\to\infty]{(d)} \frac{1}{\tau_1}\sum_{s\leqslant\tau_1} f(\Delta_s). \tag{9}$$

We now check that the expectations converge as well. First, by Proposition 3.8,

$$\mathbb{E}\left[\frac{1}{\frac{\zeta_n-1}{n^\alpha}}\sum_{i=1}^{\zeta_n} f\left(\frac{X_i}{n}\right)\right] = n^{1+\alpha}\mathbb{E}\left[f\left(\frac{X_1}{n}\right)\frac{1}{n+X_1}\right].$$

Recall that $[x]$ denotes the integer part of x. To estimate the right-hand side, using the explicit expression of f, write

$$\begin{aligned} n^\alpha\mathbb{E}\left[f\left(\frac{X_1}{n}\right)\frac{n}{n+X_1}\right] &= \sum_{k=an}^{\infty}\mathbb{P}(X_1 = k)\frac{n^{\alpha+1}}{n+k} \\ &= \int_{[an]}^{\infty} ds\,\mathbb{P}(X_1 = [s])\frac{n^{\alpha+1}}{n+[s]} \\ &= \int_{[an]/n}^{\infty} du\,\mathbb{P}(X_1 = [un])\frac{n^{\alpha+1}}{1+\frac{[un]}{n}}. \end{aligned}$$

Thefore, by dominated convergence,

$$n^\alpha\mathbb{E}\left[f\left(\frac{X_1}{n}\right)\frac{k}{k+X_1}\right] \xrightarrow[n\to\infty]{} \int_a^\infty \frac{1}{1+u}\cdot\frac{c}{u^{1+\alpha}}du = \int_0^\infty \frac{f(u)}{1+u}\,\Pi(du).$$

An extension of Proposition 3.8 and similar arguments show that

$$\mathbb{E}\left[\left(\frac{1}{\frac{\zeta_n-1}{n^\alpha}}\sum_{i=1}^{\zeta_n} f\left(\frac{X_i}{n}\right)\right)^2\right]$$

is bounded (and actually converges) as $n \to \infty$ (we leave the details to the reader). Therefore, by uniform integrability, the convergence (9) also holds in expectation, and the desired result follows. □

References

[1] J. BENNIES AND G. KERSTING, *A random walk approach to Galton-Watson trees*, J. Theoret. Probab., 13 (2000), pp. 777–803.

[2] J. BERTOIN, *On the maximal offspring in a critical branching process with infinite variance*, J. Appl. Probab., 48 (2011), pp. 576–582.

[3] J. BIENAYMÉ, *De la loi de multiplication et de la durée des familles: probabilités*,, Société philomathique de Paris, (1845).

[4] L. CHEN, N. CURIEN, AND P. MAILLARD, *In preparation*.

[5] T. E. HARRIS, *First passage and recurrence distributions*, Trans. Amer. Math. Soc., 73 (1952), pp. 471–486.

[6] J. JACOD AND A. N. SHIRYAEV, *Limit theorems for stochastic processes*, vol. 288 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, second ed., 2003.

[7] O. KALLENBERG, *Foundations of modern probability*, Probability and its Applications (New York), Springer-Verlag, New York, second ed., 2002.

[8] A. E. KYPRIANOU, *Introductory lectures on fluctuations of Lévy processes with applications*, Universitext, Springer-Verlag, Berlin, 2006.

[9] J.-F. LE GALL, *Random trees and applications*, Probability Surveys, (2005).

[10] J.-F. LE GALL AND Y. LE JAN, *Branching processes in Lévy processes: the exploration process*, Ann. Probab., 26 (1998), pp. 213–252.

[11] H. W. WATSON AND F. GALTON, *On the probability of the extinction of families.*, The Journal of the Anthropological Institute of Great Britain and Ireland, 4 (1875), pp. 138–144.