Week 16 (Final exam): Monday, January 29th, 13:15-15:15 pm

Very important:

– Please use different sheets of paper for different parts (or, in other words, use a new sheet of paper if you change parts).

– Please write your name on the sheets of paper.

All the exercises are independent. You may treat them in any order you want. The quality, the precision and the presentation of your mathematical writing will play a role in the appreciation of your work.

Part 1

Exercise 1. 1) Let $(\Omega, \mathbb{P})$ be a finite probability space. Show that for every $A, B \subset \Omega$, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Remark. You may use without proof the fact that if $(A_i)_{1 \leq i \leq n}$ are pairwise disjoint events, then $\mathbb{P}(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mathbb{P}(A_i)$.

2) Give the definition of the signature of a permutation. What ways do you know to compute the signature of a permutation?

Exercise 2. 1) Give an example of sets E, F and an assertion $P(x,y)$ (with $x \in E$ and $y \in F$) such that the assertion "$\forall x \in E,\ \exists y \in F,\ P(x,y)$" is true but the assertion "$\exists x \in E,\ \forall y \in F,\ P(x,y)$" is false. Justify your answer.

2) Can one find sets E, F and an assertion $P(x,y)$ (with $x \in E$ and $y \in F$) such that the assertion "$\exists x \in E,\ \forall y \in F, P(x,y)$" is true but the assertion "$\forall x \in E,\ \exists y \in F, P(x,y)$" is false? Justify your answer.

Part 2

Exercise 3. Let A and B be two subsets of a set E.

1) Show (carefully) that $A \subset B \iff A \cup B = B$.

2) Show (carefully) that $A = B \iff A \cap B = A \cup B$.

Exercise 4. 1) Let $\mathcal{S} = \{(u_n)_{n \geq 0}\}$ be the set of all sequences of real numbers. Let $F : \mathcal{S} \to \mathcal{S}$ be the function defined by $F : (u_0, u_1, u_2, u_3 \ldots) \mapsto (u_0, u_2, u_4, u_6, u_8, \ldots)$. Is F one-to-one? Onto? Justify your answers.

2) Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by $G : (a,b,c) \mapsto (2b, -c, a/2)$. Show that G is a bijection and give the expression of G^{-1}. Justify your answer.

Part 3

Exercise 5. Fix $p \in [0,1]$. Sophia flips a coin three times in a row. We assume that the coin is unfair: it gives 'Heads' with probability p, and 'Tails' with probability $1-p$. We assume that we are given a finite probability space $(\Omega, \mathbb{P})$ with, for $1 \leq i \leq 3$, events $H_i = \{\text{the } i\text{-th result is 'Heads'}\}$ such that $\mathbb{P}(H_i) = p$ for $1 \leq i \leq 3$ and such that H_1, H_2, H_3 are independent.

1) For each one of the following events, write the event using only symbols $H_1, H_2, H_3, \overline{H_1}, \overline{H_2}, \overline{H_3}, \cap, \cup$:

a) $E = \{\text{None of the result is 'Heads'}\}$ b) $F = \{\text{(At least) the two first results are identical}\}$

c) $G = \{\text{We obtain (at least) two consecutive 'Heads'}\}$.

2) Compute $\mathbb{P}(E)$, $\mathbb{P}(F)$, $\mathbb{P}(G)$.

Part 4

Exercise 6. Let $n \geq 1$ be an integer.

1) (example) For $j = 2$ and $n = 3$, check that $\sum_{k=j}^{n} \binom{k-1}{j-1} = \binom{n}{j}$

2) Let P_n be the property "For every integer $1 \leq j \leq n$, $\sum_{k=j}^{n} \binom{k-1}{j-1} = \binom{n}{j}$". Show that P_n is true by induction.

We denote by S_n the set of all permutations of $\{1, 2, \ldots, n\}$ and we say that a permutation $\sigma \in S_n$ has a record at $j \in \{1, \ldots, n\}$ if $\sigma(i) < \sigma(j)$ for every positive integer i such that $i < j$.

3) Fix two positive integers j, k such that $1 \leq j \leq k \leq n$. Justify that there are $\binom{k-1}{j-1}(j-1)!(n-j)!$ permutations $\sigma \in S_n$ such that both $\sigma(j) = k$ and σ has a record at j.

4) Let $\mathbb{P}$ be the uniform probability on S_n. For $1 \leq j \leq n$, give a simple expression of the probability of the event $\{\sigma \in S_n : \sigma \text{ has a record at } j\}$.

Hint. If you do not manage to solve one of the previous questions, you can assume that the results of the previous questions are true in order to solve this question.

Part 5 (optional)

This part is optional and does not count in the grading. Please go beyond only if you have solved all the previous exercises.

Exercise 7. Let $n \geq 2$ be an integer. We say that a permutation σ of $\{1, 2, \ldots, n\}$ is decomposable if there exists an integer $1 \leq k \leq n-1$ such that $\sigma(\{1, 2, \ldots, k\}) = \{1, 2, \ldots, k\}$. Denote by d_n the number of decomposable permutations σ of $\{1, 2, \ldots, n\}$ and let $p_n = \frac{d_n}{n!}$ the probability that permutation of S_n chosen uniformly at random is decomposable. Find an asymptotic equivalent of p_n when $n \to \infty$ (that is, the simplest expression as possible a_n such that $p_n/a_n \to 1$ when $n \to \infty$).

Exercise 8. Let $n \geq 1$ be an integer. An urn contains n blue balls and n red balls. Choose successively without replacement balls at random until there is only one colour left in the urn. Denote by H_n the number of balls that then remain in the urn. For every $0 \leq k \leq n$, compute $\mathbb{P}(H_n = k)$.

You can express your result using binomial coefficients.

Exercise 9. What does the following image represent, and how was it obtained?