Week 9 (Midterm exam): Tuesday, November 28th, 8am-10am

Very important:

– Please use different sheets of paper for different parts (or, in other words, use a new sheet of paper if you change parts).

– Please write your name on the sheets of paper.

All the exercises are independent. You may treat them in any order you want. The quality, the precision and the presentation of your mathematical writing will play a role in the appreciation of your work.

Part 1

Exercise 1.

1) Give an example of a function which is not onto and an example of a function which is not one-to-one.

2) Show that if $f : E \to F$ is a one-to-one function and A is a subset of E, then $A = f^{-1}(f(A))$.

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x|$ for every $x \in \mathbb{R}$.

1) Give a simple expression of the set $f([-2, 2])$. Justify your answer.

2) Give a simple expression of the set $f^{-1}([-1, 2])$. Justify your answer.

Part 2

Exercise 3. Let P be the following mathematical assertion

P : "There exists a real number x such that for every real number y we have $x + y = y$".

1) Write P using quantifiers.

2) Write the negation of P using quantifiers.

3) Is P true? Justify your answer.

Exercise 4. Fix two integers $a, b \in \mathbb{Z}$. Show that

$$\text{if } a^2(b^2 - b) \text{ is odd, then } a, b \text{ are both odd.}$$

Hint. You may use the method of proof by contrapositive.

Part 3

Exercise 5.

1) Show that for every integer $n \geq 1$ we have $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

2) How many integers $1 \leq a, b, c, d \leq 100$ such that $a < b$ and $a < c$ and $a < d$ are there?

Hint. If you do not manage to solve the first question, you can assume that the result of the first question is true in order to solve this second question.

Part 4

Exercise 6. Let g be the function defined by

$$\begin{array}{cccc} g & : & \mathbb{R}\times\mathbb{R} & \to & \mathbb{R}\times\mathbb{R} \\ & & (x,y) & \mapsto & (x+y, xy) \end{array}$$

1) Is g one-to-one? Justify your answer.

2) Is g onto? Justify your answer.

Part 5 (optional)

This part is optional and does not count in the grading. Please go beyond only if you have solved all the previous exercises.

Exercise 7. Fix an integer $n \geq 2$. In how many ways can we choose two subsets A and B of $\{1, 2, \ldots, n\}$ such that $A \subset B$?

Exercise 8. If A is a set, recall that $\mathcal{P}(A)$ denotes the set of all subsets of A. Let $f : E \to F$ be a function. Set $\mathcal{S} = \{X \subset E : f^{-1}(f(X)) = X\}$. Define the function g as follows:

$$\begin{array}{ccccc} g & : & \mathcal{S} & \to & \mathcal{P}(f(E)) \\ & & A & \mapsto & f(A) \end{array}$$

Show that g is a bijection.

Exercise 9. What does the following image represent?