Week 12: Finite probability spaces

Instructor: Igor Kortchemski (igor.kortchemski@polytechnique.edu)
Tutorial Assistants:
- Apolline Louvet (groups A&B, apolline.louvet@polytechnique.edu)
- Milica Tomasevic (groups C&E, milica.tomasevic@polytechnique.edu)
- Benoît Tran (groups D&F, benoit.tran@polytechnique.edu).

"The theory of probability, as a mathematical discipline, can and should be developed from axioms in exactly the same way as Geometry and Algebra. This means that after we have defined the elements to be studied and their basic relations, and have stated the axioms by which these relations are to be governed, all further exposition must be based exclusively on these axioms, independent of the usual concrete meaning of these elements and their relations. "

Andreï Kolmogorov (1956).

Kolmogorov explains that the mathematical theory of probability is developed independent of all the real world restrictions, only being subject to the constraints of logic. *The misconception to think that a mathematical discipline automatically deals with something real has led to unrealistic models.*

1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Fix integers $1 \le r \le n$. We put r balls (numbered from 1 to r) into n urns (numbered from 1 to n), uniformly at random.

1) Construct a finite probability space to model this experiment.

2) Find the probability that every urn has at most one ball.

3) Find the probability that there is an urn having at least two balls.

Solution of exercise 1.

1) For every ball, we record in which urn it is put. In other words, we take

$$\Omega = \{1, 2, \dots, n\}^r.$$

If $\omega = (\omega_1, \dots, \omega_r) \in \Omega$, the integer ω_i represents the number of the urn in which falls the ball i. Note that $\mathrm{Card}(\Omega) = n^r$ (because for every one of the r balls we have n possible choices).

We equip Ω with the uniform probability measure, which we denote by $\mathbb{P}$.

2) Let A be the event "every urn has at most one ball". By definition of the uniform probability measure, $\mathbb{P}(A) = \frac{\mathrm{Card}(A)}{\mathrm{Card}(\Omega)}$. To find $\mathrm{Card}(A)$, we count the number of configurations such that every urn has at most one ball. The idea is take the viewpoint of the balls (and not of the urns): for the first ball, we have n choices for its urn. For the second ball, we have $n-1$ choices for its urn, and so on. Therefore $\mathrm{Card}(A) = n(n-1)\cdots(n-r+1)$, so that

$$\mathbb{P}(A) = \frac{n(n-1)\cdots(n-r+1)}{n^r}.$$

3) Let B be the event "there is an urn having at least two balls". Since $\overline{A} = B$, we have $\mathbb{P}(B) = 1 - \mathbb{P}(A)$. □

Exercise 2. Let A, B, C be events of a finite probability space. Using set operations, write the following events:

(a) A is not realized

(b) None of the events A, B nor C are realized.

(c) Only one of the events A, B or C is realized.

(d) At least two of the events A, B, C are realized.

(e) No more that two of the events A, B, C are realized

Solution of exercise 2.

Recall that "blabla is realized" is a probabilistic formulation for the set $\{\omega \in \Omega : \omega \in \text{blabla}\}$.

(a) $\overline{A}$.

(b) $\overline{A} \cap \overline{B} \cap \overline{C}$.

(c)

$$\left(A \cap \overline{B} \cap \overline{C}\right) \cup \left(\overline{A} \cap B \cap \overline{C}\right) \cup \left(\overline{A} \cap \overline{B} \cap C\right).$$

(d)

$$\left(A \cap B\right) \cup \left(B \cap C\right) \cup \left(A \cap C\right)$$

(e) This means that at least one of the events A, B, C is not realized:

$$\overline{A} \cup \overline{B} \cup \overline{C}.$$

□

Exercise 3. Consider a parking lot having 8 consecutive slots (meaning that the slots are one after the other). A blue car and a red car have parked uniformly at random.

1) Construct a finite probability space to model this experiment.

2) What is the probability that the first slot has been taken by a car?

3) What is the probability that the two cars have parked next to each other?

Solution of exercise 3.

1) We number the slots $1, 2, 3, 4, 5, 6, 7, 8$ and take

$$\Omega = \{(i, j) \in \{1, 2, 3, 4, 5, 6, 7, 8\}^2 : i \neq j\}.$$

The first integer i represents the slot taken by the blue car and the second integer j represents the slot taken by the red car (the condition $i \neq j$ means that the two cars cannot park at the same place). We take the uniform probability on Ω.

We have $\mathrm{Card}(\Omega) = 8^2 - 8 = 56$ (because $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}^2 \backslash \{(i, j) \in \{1, 2, 3, 4, 5, 6, 7, 8\}^2 : i = j\}$)

2) Let A be the event "the first slot has been taken by a car". We have

$$A = \{(1, 2), (1, 3), \ldots, (1, 8), (2, 1), (3, 1), \ldots, (8, 1)\}$$

so that $\mathrm{Card}(A) = 14$. Therefore the probability that the first slot has been taken by a car is $\frac{14}{56} = \frac{1}{4}$.

3) Let B be the event that the two cars have parked next to each other. To count the number of elements of B, observe that such a configuration can be coded by the position of the first taken slot (7 possibilities) and then by choosing the car that takes this slot (2 choices). By the multiplicative rule, we get 14 possibilities. Hence the probability that the two cars have parked next to each other is $\frac{14}{56} = \frac{1}{4}$. □

Exercise 4. Let $(\Omega, \mathbb{P})$ be a finite probability space.

1) If A, B are events show that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Hint: write $A \cup B = (A \backslash A \cap B) \cup (A \cap B) \cup (B \backslash A \cap B)$.

2) Let $n \geq 2$ be an integer. Let $(A_k)_{1 \leq k \leq n}$ be a sequence of events. Show that $\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} \mathbb{P}(A_k)$.

3) *(Application)* Let $n \geq 2$ be an integer.

a) Let $(A_k)_{1 \leq k \leq n}$ be a sequence of events such that $\mathbb{P}(A_k) = 0$ for every $1 \leq k \leq n$. Show that $\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = 0$.

b) Let $(A_k)_{1 \leq k \leq n}$ be a sequence of events such that $\mathbb{P}(A_k) = 1$ for every $1 \leq k \leq n$. Show that $\mathbb{P}\left(\bigcap_{k=1}^{n} A_k\right) = 1$.

Solution of exercise 4.

1) We write $A \cup B = (A \backslash A \cap B) \cup (A \cap B) \cup (B \backslash A \cap B)$ and observe that this is a union of pairwise disjoint events. Hence, by the result stated in the lecture (namely: if $n \geq 2$ and $(A_k)_{1 \leq k \leq n}$ are pairwise disjoint events; then $\mathbb{P}(\cup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \mathbb{P}(A_k)$) and whose proof is available on the course webpage,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \backslash A \cap B) + \mathbb{P}(A \cap B) + \mathbb{P}(B \backslash A \cap B).$$

As was seen in the lecture, we have $\mathbb{P}(A) = \mathbb{P}(A \backslash B) - \mathbb{P}(A \cap B)$ and $\mathbb{P}(B) = \mathbb{P}(B \backslash A) - \mathbb{P}(A \cap B)$. Therefore

$$\mathbb{P}(A \cup B) = (\mathbb{P}(A) - \mathbb{P}(A \cap B)) + \mathbb{P}(A \cap B) + (\mathbb{P}(B) - \mathbb{P}(A \cap B)),$$

and the desired result follows.

2) We argue by induction. Let $P(n)$ be the property:

$$\text{“If } (A_k)_{1 \leq k \leq n} \text{ are events, then } \mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} \mathbb{P}(A_k)\text{”.}$$

Basis step. For $n = 2$, by the previous question,

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2).$$

Since $\mathbb{P}(A_1 \cap A_2) \geq 0$, we get that $\mathbb{P}(A_1 \cup A_2) \leq \mathbb{P}(A_1) + \mathbb{P}(A_2)$.

Inductive step. Fix $n \geq 2$ and assume that $P(n)$ is true. Let $(A_k)_{1 \leq k \leq n+1}$ be events. Using the fact that $P(n)$ and $P(2)$ are true, we have

$$\begin{aligned}\mathbb{P}\Big(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}\Big) &= \mathbb{P}\Big(A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup (A_n \cup A_{n+1})\Big) \\ &\leq \mathbb{P}(A_1) + \cdots + \mathbb{P}(A_{n-1}) + \mathbb{P}(A_n \cup A_{n+1}) \\ &\leq \mathbb{P}(A_1) + \cdots + \mathbb{P}(A_{n-1}) + \mathbb{P}(A_n) + \mathbb{P}(A_{n+1})\end{aligned}$$

This completes the proof.

3)

a) Using the previous question, write

$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} \mathbb{P}(A_k) = \sum_{k=1}^{n} 0 = 0.$$

Hence $\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = 0$.

b) We use the equality

$$\overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \overline{A_k}$$

(which can be checked by double inclusion) to write

$$\mathbb{P}\left(\bigcap_{k=1}^{n} A_k\right) = 1 - \mathbb{P}\left(\overline{\bigcap_{k=1}^{n} A_k}\right) = 1 - \mathbb{P}\left(\bigcup_{k=1}^{n} \overline{A_k}\right).$$

But $\mathbb{P}\left(\overline{A_k}\right) = 1 - \mathbb{P}(A_k) = 0$ for every $1 \leq k \leq n$, so $\mathbb{P}\left(\bigcup_{k=1}^{n} \overline{A_k}\right) = 0$ by a), which completes the proof. □

2 Homework exercise

You have to individually hand in the written solution of the next exercise to your TA on Monday, January 6th.

Exercise 5. Five people each throw a fair dice (a dice is fair when the probability of falling on its different faces is the same). Among the five people, three people have one dice with 6 faces from 1 to 6 and two people have one dice with 4 faces from 1 to 4.

1) Construct a finite probability space to model this experiment.

2) Compute the probabilities of the following events:

(a) everyone gets 4; (b) everyone gets 5 (c) all the numbers are different;

(d) at least two people obtain the same number; (e) one of the dices having 6 faces gives the same number as one of the dices having 4 faces.

Solution of exercise 5. 1) We take $\Omega = \{1,2,3,4,5,6\}^3 \times \{1,2,3,4\}^2$ (which has $6^3 \times 4^2$ elements), equipped with the uniform probability measure (recall that this means that if $A \subseteq \Omega$, $\mathbb{P}(A) = \frac{\mathrm{Card}(A)}{\mathrm{Card}(\Omega)}$).

2)

(a) $\mathbb{P}(\text{everyone gets 4}) = \frac{1}{6^3 \times 4^2} \left(= \frac{1}{3456}\right)$.

(b) The event “everyone gets 5 ” is the empty set, so its probability is 0.

(c) We count configurations such that all the numbers are different. To this end, the idea is to start with the 4-faced dices: 4 choices for the outcome of the first dice, then 3 choices for the outcome of the second dice, then 4 choices for the outcome of the first 6-faced dice, 3 choices for the second and 2 choices for the third one. Therefore

$$\mathbb{P}(\text{all the numbers are different}) = \frac{4 \times 3 \times 4 \times 3 \times 2}{6^3 \times 4^2} \left(= \frac{1}{12} \right).$$

(d) The complementary event is ”all the numbers are different”, so the answer is

$$1 - \frac{4 \times 3 \times 4 \times 3 \times 2}{6^3 \times 4^2} \left(= \frac{11}{12} \right).$$

(e) The complementary event is ”the numbers appearing on the dices having 6 faces are all different from the numbers appearing on the dices having 4 faces”. We count such configurations :

First case: the numbers appearing on the dices having 4 faces are the same; 4 possibilities. For each one of these possibilities, we have 5^3 possibilities for the 6-faced dices. In total this gives 4×5^3 possibilities.

Second case: the numbers appearing on the dices having 4 faces are different; 4×3 possibilities. For each one of these possibilities, we have 4^3 possibilities for the 6-faced dices. In total this gives $4 \times 3 \times 4^3$ possibilities.

We conclude that the result is

$$1 - \frac{4 \times 5^3 + 4 \times 3 \times 4^3}{6^3 \times 4^2} \left(= 1 - \frac{317}{864} = \frac{547}{864} \right).$$

□

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 12.

Exercise 6. Fix integers $1 \leq r \leq n$. We put r (indistinguishable) purple balls into n urns (numbered from 1 to n), uniformly at random.

1) Construct a finite probability space to model this experiment.

2) Find the probability that every urn has at most one ball.

Solution of exercise 6.

1) For every urn, we record the number of balls it contains. In other words, we take

$$\Omega = \{(a_1, \ldots, a_n) : \text{ for every } 1 \leq i \leq n, 0 \leq a_i \leq r \text{ and } a_1 + \cdots + a_n = r\}.$$

If $\omega = (a_1, \ldots, a_n) \in \Omega$, the integer a_i represents the number of balls which fall in the urn i. Recall from Exercise 10 of week 9 that $\mathrm{Card}(\Omega) = \binom{n+r-1}{r}$.

We equip Ω with the uniform probability measure, which we denote by $\mathbb{P}$.

2) Let A be the event "every urn has at most one ball". Note that constructing such a configuration amounts to choosing the r urns which will each contain a ball. Therefore $\mathrm{Card}(A) = \binom{n}{r}$, so that

$$\mathbb{P}(A) = \frac{\binom{n}{r}}{\binom{n+r-1}{r}}.$$

□

Exercise 7. Let $A_1, \ldots, A_n$ be events of a finite probability space $(\Omega, \mathbb{P})$.

1) Show that $\mathbb{P}(A_1 \cap \cdots \cap A_n) \leq \min_{1 \leq i \leq n} \mathbb{P}(A_i)$.

2) Show that $\mathbb{P}(A_1 \cap \cdots \cap A_n) \geq \sum_{i=1}^{n} \mathbb{P}(A_i) - (n-1)$.

Solution of exercise 7.

1. For every $i \in \{1, \ldots, n\}$, $A_1 \cap \cdots \cap A_n \subseteq A_i$, hence $\mathbb{P}(A_1 \cap \cdots \cap A_n) \leq \mathbb{P}(A_i)$. Therefore $P(A_1 \cap \cdots \cap A_n) \leq \min_{1 \leq i \leq n} \mathbb{P}(A_i)$.

2. Let us consider the complementary event:

$$\begin{aligned} 1 - \mathbb{P}(A_1 \cap \cdots \cap A_n) &= \mathbb{P}(\overline{A}_1 \cup \cdots \cup \overline{A}_n) \\ &\leq \sum_{i=1}^{n} \mathbb{P}(\overline{A_i}) \\ &= \sum_{i=1}^{n} (1 - \mathbb{P}(A_i)) = n - \sum_{i=1}^{n} \mathbb{P}(A_i), \end{aligned}$$

and the desired result follows.

□

Exercise 8. Let $(\Omega, \mathbb{P})$ be a finite probability space, $n \geq 1$ an integer and $A_1, \ldots, A_n$ events. Set $[\![1, n]\!] = \{1, 2, \ldots, n\}$. Show that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\substack{I \subseteq [\![1,n]\!] \\ I \neq \emptyset}} (-1)^{-1+|I|}\, \mathbb{P}\left(\bigcap_{i \in I} A_i\right).$$

Solution of exercise 8. We argue by induction on n. For $n = 1$, there is nothing to do.

Assume that the result holds for a fixed $n \geq 1$ and let us show that it holds for $n+1$. Let

$A_1, \dots, A_{n+1}$ be events. Set $\widehat{A}_n = A_1 \cup A_2 \cup \cdots \cup A_n$. Then,

$$\begin{aligned}
\mathbb{P}\Big(\widehat{A}_n \cup A_{n+1}\Big) &= \mathbb{P}(\widehat{A}_n) + \mathbb{P}(A_{n+1}) - \mathbb{P}(\widehat{A}_n \cap A_{n+1}) \\
&= \mathbb{P}(\widehat{A}_n) + \mathbb{P}(A_{n+1}) - \mathbb{P}\Big(\bigcup_{i=1}^{n} (A_i \cap A_{n+1})\Big) \\
&= \sum_{\substack{I \subseteq [\![1,n]\!] \\ I \neq \varnothing}} (-1)^{-1+|I|}\, \mathbb{P}\Big(\bigcap_{i \in I} A_i\Big) + \mathbb{P}(A_{n+1}) - \sum_{\substack{I \subseteq [\![1,n]\!] \\ I \neq \varnothing}} (-1)^{-1+|I|}\, \mathbb{P}\Big(\bigcap_{i \in I} (A_i \cap A_{n+1})\Big) \\
&= \sum_{\substack{I \subseteq [\![1,n+1]\!] \\ I \neq \varnothing, n+1 \notin I}} (-1)^{-1+|I|}\, \mathbb{P}\Big(\bigcap_{i \in I} A_i\Big) + \mathbb{P}(A_{n+1}) + \sum_{\substack{I \subseteq [\![1,n+1]\!] \\ I \neq \{n+1\}, n+1 \in I}} (-1)^{-1+|I|}\, \mathbb{P}\Big(\bigcap_{i \in I} A_i\Big) \\
&= \sum_{\substack{I \subseteq [\![1,n+1]\!] \\ I \neq \varnothing}} (-1)^{-1+|I|}\, \mathbb{P}\Big(\bigcap_{i \in I} A_i\Big),
\end{aligned}$$

and the proof is complete.

Remark. This formula, called the inclusion-exclusion formula, extends the identity $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ to a finite number of events, and is sometimes very useful to compute the probability of a union of non-disjoint events. □

Exercise 9. Let $(\Omega, \mathbb{P})$ be a finite probability space and fix two events A, B. Show that $|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq \frac{1}{4}$.

Solution of exercise 9. ▹ First observe that $\mathbb{P}(A) \geq \mathbb{P}(A \cap B)$ and $\mathbb{P}(B) \geq \mathbb{P}(A \cap B)$; hence

$$\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B) \leq \mathbb{P}(A \cap B) - \mathbb{P}(A \cap B)^2 \leq \frac{1}{4}$$

by studying the variations of $x \mapsto x(1-x)$ on $[0,1]$.

▹ We now show that $\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B) \geq -\frac{1}{4}$.

- First case: assume that $\mathbb{P}(A) + \mathbb{P}(B) < 1$ (that is $\mathbb{P}(B) < 1 - \mathbb{P}(A)$). Then,

$$\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B) \geq 0 - \mathbb{P}(A)\mathbb{P}(B) > -\mathbb{P}(A)(1 - \mathbb{P}(A)) \geq -\frac{1}{4}$$

  by using the same argument with the function $x \mapsto x(1-x)$.

- Second case: assume that $1 - \mathbb{P}(B) \leq \mathbb{P}(A)$. Then,

$$\begin{aligned}
\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) - \mathbb{P}(A)\mathbb{P}(B) \\
&\geq \mathbb{P}(A) + \mathbb{P}(B) - 1 - \mathbb{P}(A)\mathbb{P}(B) \\
&= -(1 - \mathbb{P}(A))(1 - \mathbb{P}(B)) \\
&\geq -(1 - \mathbb{P}(A))\mathbb{P}(A) \geq -\frac{1}{4}.
\end{aligned}$$

□

4 Fun exercise (optional)

The solution of this exercise will be available on the course webpage at the end of week 12.

Exercise 10. A colony of n vampires lives in the Carpathian mountains. The Countess Dracula wishes to estimate the number n of vampires (Dracula is not considered as being part of the colony). To do so, one night, she captures ten of them at random, bits their ears and releases them. The next night, she captures again 10 vampires at random. It turns out that 3 of them have their ears bitten.

Denote by p_n the probability that 3 vampires have their ears bitten when one chooses 10 of them uniformly at random in a population of n vampires with 10 having their ears bitten. For what integer n is the quantity p_n maximal?

Hint. You may simplify the quantity $\frac{p_n}{p_{n+1}}$.

Solution of exercise 10. The number of ways of choosing 10 vampires among n is $\binom{n}{10}$. Now, choosing 10 vampires in a population of n vampires with 10 having their ears bitten so that only 3 have their ears bitten amounts to first choosing 3 vampires among the 10 with bitten ears, and then 7 vampires among the $n-10$ others without bitten ears. Therefore

$$p_n = \frac{\binom{10}{3}\binom{n-10}{7}}{\binom{n}{10}}.$$

To find the value of n which maximises this quantity, we note that

$$\frac{p_n}{p_{n+1}} = \frac{n^2-15n-16}{n^2-18n+81},$$

which is (strictly) less that 1 for $n \leq 32$ and (strictly) greater than 1 for $n \geq 33$. Thus p_n is maximal for $n = 33$.

Remark. In the language of Statistics, we have built an estimator based on the maximum likelihood for the a priori unknown quantity n. The method used in this exercise is simple, but is used by biologists studying populations of animals (capture-mark-recapture method, see https://en.wikipedia.org/wiki/Mark_and_recapture) □