Week 8: Cardinality and combinatorics

Instructor: Igor Kortchemski (igor.kortchemski@polytechnique.edu)
Tutorial Assistants:
- Apolline Louvet (groups $A\&B$, apolline.louvet@polytechnique.edu)
- Milica Tomasevic (groups $C\&E$, milica.tomasevic@polytechnique.edu)
- Benoît Tran (groups $D\&F$, benoit.tran@polytechnique.edu).

1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. How many integers $1 \leq a, b, c \leq 100$ such that $a < b$ and $a < c$ are there?

Solution of exercise 1. For a given choice of a, there are $(100-a)^2$ choices of (b,c). The total number of choices is therefore

$$\sum_{a=1}^{100}(100-a)^2 = \sum_{a=0}^{99} a^2 = \sum_{a=1}^{99} a^2 = \frac{99 \cdot 100 \cdot 199}{6} \qquad \Big(= 328350 \Big)$$

by using the formula $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ which can be shown by induction.

More formal solution. Set $E = \{(a,b,c) : 1 \leq a,b,c, \leq 100, a < b \text{ and } a < c\}$, and for $1 \leq i \leq 100$, write $E_i = \{(i,b,c) : 1 \leq b,c \leq 100, i < b \text{ and } i < c\}$. Then $E = \cup_{i=1}^{100} E_i$ and the union is disjoint. Therefore

$$\#E = \sum_{i=1}^{100} \#E_i$$

and $\#E_i = (100-i)^2$. □

Exercise 2. In how many ways is it possible to arrange in a line 7 girls and 3 boys in the following cases:

1) When the 3 boys follow each other.

2) When the first and last person are girls, and when all the 3 boys do not follow each other.

Solution of exercise 2.

1) First consider the boys as one person. Then there are 8! possibilities (8 possibilites for the first person, then 7 for the second one, etc.). Then one has to choose the order of the boys: 3! possibilities. Thefore the result is $3! \cdot 8!$.

2) The idea is to use the "complement rule". Let A be the set of configurations where the first and the last person are girls. Let B be the set of configurations where the first and the last person are girls and when boys do not follow each other. Then $\#(A \setminus B) = \#A - \#B$, and $A \setminus B$ represents the set of configurations where the first and the last person are girls and the boys follow each other. We have,

$$\#A = 7 \cdot 6 \cdot 8!$$

(7 possibilities of choosing a girl for the first position, 6 for the last position, $10-2 = 8$ possibilities

for the second position, 7 for the third one and so on) and a similar argument as for the first question gives

$$\#(A \setminus B) = 7 \cdot 6 \cdot 6! \cdot 3!$$

The result is therefore

$$\#B = \#A - \#(A \setminus B) = 7 \cdot 6 \cdot 8! - 7 \cdot 6 \cdot 6! \cdot 3! = 7!(6 \cdot 7 \cdot 8 - 6 \cdot 3!) = 300 \cdot 7!$$

□

Exercise 3. Let $n \geq 2$ be an integer, and set $E = \{1, 2, \ldots, n\}$. Find the cardinalities of the following sets:

$$F = \{(i,j) \in E^2\}, \quad G = \{(i,j) \in E^2, i \neq j\}, \quad H = \{(i,j) \in E^2, i < j\}, \quad I = \{A \subseteq E, \mathrm{Card}(A) = 2\}.$$

Solution of exercise 3. faire comprendre aux élèves la différence entre l'ensemble G et l'ensemble I.

- *Intuitive version:* first n choices for i, then n choices for j.

  Formal version: by the course we have $\#F = \#E \times \#E = n^2$.

- *First solution.* n choices for i, then $n-1$ choices for j, which gives $n(n-1)$.

  Second solution. We use the "complement rule" by noticing that

$$G = \{(i,j) \in E^2\} \setminus \{(i,i), i \in E\}.$$

  Therefore

$$\#G = \#\{(i,j) \in E^2\} - \#\{(i,i), i \in E\} = n^2 - n = n(n-1).$$

- *First solution.* For a fixed $1 \leq i \leq n$, there are $n-i$ choices for j. Therefore

$$\#H = \sum_{i=1}^{n} (n-i) = \sum_{i=0}^{n-1} i = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}.$$

  Second solution. Let us define a map

$$\begin{array}{rcl} \phi : G & \longrightarrow & H \\ (i,j) & \longmapsto & \begin{cases} (i,j) & \text{if } i < j \\ (j,i) & \text{if } i > j \end{cases}. \end{array}$$

  Every pair (a,b) in H has exactly two preimages by f: (a,b) and (b,a). Hence $\#G = 2 \times \#H$ and

$$\#H = n(n-1)/2.$$

- The map

$$\psi : I \longrightarrow H, \quad A \longmapsto (\min(A), \max(A))$$

is a bijection, since it is one-to-one and onto.

Therefore $\#I = \#H = n(n-1)/2$.

□

Exercise 4. How many onto functions from $\{1,2,\ldots,n\}$ to $\{1,2,3\}$ are there?

Solution of exercise 4. We use the complement rule and find the number of functions which are not onto. First, there are 3^n functions from $\{1,2,\ldots,n\}$ to $\{1,2,3\}$. Let $f : \{1,2,\ldots,n\} \to \{1,2,3\}$ be a function which is not onto.

▹ in the case where the range of f has cardinality 1: we have 3 choices.

▹ in the case where the range of f has cardinality 2: we have 3 choices to choose the element which is not in the range of f. Then choosing the elements of $\{1,2,\ldots,n\}$ which are mapped to the smallest element of the range of f amounts to choosing a nonempty subset of $\{1,2,\ldots,n\}$ which is not $\{1,2,\ldots,n\}$ itself, which gives $2^n - 2$ choices. In total, this gives $3(2^n - 2)$ choices.

Therefore the number of onto functions from $\{1,2,\ldots,n\}$ to $\{1,2,3\}$ is

$$3^n - (3 + 3(2^n - 2)) = 3^n - 3 \cdot 2^n + 3.$$

□

Exercise 5. Let E and F be finite sets *having the same cardinality*, and let $f : E \to F$ be a function. Show that the following three assertions are equivalent:

(1) f is onto;

(2) f is one-to-one;

(3) f is a bijection.

Solution of exercise 5. Assume that $n = \#E = \#F$.

It is clear that (3) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (2). Let us first show that (1) $\Longrightarrow$ (3). Assume that f is onto, and let us show that f is one-to-one. Argue by contradiction and assume that f is not one-to-one. Then $\#f(E) < n$. Since f is onto, we have $f(E) = F$, so that $\#f(E) = \#F = n$. This is a contradiction. Hence (1) $\Longrightarrow$ (3).

Let us now show that (2) $\Longrightarrow$ (3). We shall use the following simple fact: if A and B are finite sets such that $A \subseteq B$ and $\#A = \#B$, then $A = B$ (to show this fact, we argue by contradiction: if $A \neq B$, since $A \subseteq B$, we can then find an element x such that $x \in B$ and $x \notin A$, so that $\#B > \#A$, which is a contradiction).

Assume that f is one-to-one. As a consequence, $\#f(E) = \#E = n$. Therefore $\#f(E) = \#F$ and we always have $f(E) \subseteq F$. By the simple fact above, it follows that $f(E) = F$, so that f is onto. Hence (2) $\Longrightarrow$ (3). □

2 Homework exercise

You have to individually hand in the written solution of the next exercise to your TA on Monday, November 25th.

Exercise 6.

1) How many three-digit numbers abc have exactly one digit equal to 9? Justify your answer.

2) How many three-digit numbers abc have the property that $a \neq b$ or $b \neq c$? Justify your answer.

3) How many three-digit numbers abc have the property that $b > c$? Justify your answer.

Note. A three-digit numbers cannot start with a "0", for instance 011 is not a three-digit number.

Solution of exercise 6.

1) We use the sum rule (disjunction of cases).

Case 1. The first digit is 9. Then we have 9 choices for b and 9 choices for c, which gives 81 choices.

Case 2. The second digit is 9. Then we have 8 choices for a and 9 choices for c, which gives 72 choices.

Case 3. The third digit is 9. Then we have 8 choices for a and 9 choices for b, which gives 72 choices.

In total, we have 225 such numbers.

2) We use the complement rule: we count the number of three-digit numbers such that $a = b$ and $b = c$. This means $a = b = c$, so there are 9 such numbers. Since there are $9 \times 10^2 = 900$ three-digit numbers, it follows that there are $900 - 9 = 891$ three-digit numbers abc having the property that $a \neq b$ or $b \neq c$.

3) For a fixed $1 \leq a \leq 9$, and a fixed $0 \leq b \leq 9$, there are b choices for c. By the sum rule, the answer is

$$\sum_{a=1}^{9} \sum_{b=0}^{9} b = 9 \sum_{b=0}^{9} b = 9 \times \frac{9 \times 10}{2} = 405.$$

□

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 8.

Exercise 7. Fix an integer $n \geq 1$ and set $E = \{1, 2, \ldots, n\}$. A function $f : E \to E$ is an *involution* if $f(f(x)) = x$ for every $x \in E$. Let u_n be the number of involutions of E.

1) Compute u_1 and u_2.

2) Show that for every $n \geq 1$, $u_{n+2} = u_{n+1} + (n+1)u_n$.

Solution of exercise 7.

1) We have $u_1 = 1$ (there is only one function from a set with one element to itself) and $u_2 = 2$. Indeed, we already saw in the course that an involution is a bijection. There are two bijections from $\{1, 2\}$ to itself (which are given by $f(1) = 1, f(2) = 2$ and $g(2) = 1$, $g(1) = 2$ and both are involutions.

2) Fix $n \geq 1$ and consider an involution $f : \{1, 2, \ldots, n+2\} \to \{1, 2, \ldots, n+2\}$. The idea is to look at

what happens to $f(1)$.

▹ If $f(1)=1$, then f, restricted to $\{2,\dots,n\}$ is an involution on a set with $n+1$ elements, which gives u_{n+1} possibilities.

▹ If $f(1)\neq 1$, then there are $n+1$ possibilities for $f(1)$. Once $f(1)$ has been chosen, we are left with an involution on a set with n elements (that is $\{1,2,\dots,n+2\}\backslash\{1,f(1)\}$). This gives

$$u_{n+2}=u_{n+1}+(n+1)u_n.$$

More formal solution. The set of all involutions on $\{1,2,\dots,n+2\}$ can be we written as a disjoint union

$$E\cup E_2\cup E_3\cup\cdots\cup E_{n+2},$$

where E is the set of all involutions f on $\{1,2,\dots,n+2\}$ such that $f(1)=1$, and for $2\leq i\leq n+2$, E_i is set of all involutions f on $\{1,2,\dots,n+2\}$ such that $f(1)=i$ and $f(i)=1$.

An element of E (which is a function) is uniquely defined by its action on $\{2,\dots,n+2\}$, which is an involution on this set of $n+1$ elements, so that $\#E=u_{n+1}$.

An element of E_i, for $1\leq i\leq n+2$, is uniquely defined by its action on $\{1,2,\dots,n+2\}\backslash\{1,i\}$, which is an involution on this set of $n+1$ elements, so that $\#E_i=u_n$.

We conclude that

$$u_{n+2}=u_{n+1}+(n+1)u_n.$$

□

Exercise 8. (Shephard lemma or black sheep lemma) Let E and F be two finite sets and $f:E\to F$ a function. Assume that there exists an integer $p\geq 1$ such that for every $y\in F$, $\#f^{-1}(\{y\})=p$. Show that $\#E=p\cdot\#F$.

Solution of exercise 8. To simplify notation, set $m=\#E$, $n=\#F$ and write $F=\{y_1,y_2,\dots,y_n\}$. For $1\leq i\leq n$, set $A_i=f^{-1}(\{y_i\})$. We claim that

$$E=\bigcup_{i=1}^{n}A_i$$

and that this union is disjoint. First, it is clear that $\cup_{i=1}^{n}A_i\subseteq E$ (since $A_i\subseteq E$ for every $1\leq i\leq n$). On the other hand, if $x\in E$, and if $f(x)=y_j$ with a certain $1\leq j\leq n$, then $x\in A_j$. The fact that the union is disjoint was established in Exercise 5 of the Tutorial Sheet 6.

Therefore

$$\#E=\sum_{i=1}^{n}\#A_i=\sum_{i=1}^{n}p=pn.$$

Remark. Can you guess why I call this lemma “shephard lemma” or “black sheep lemma”? □

Exercise 9. (Inclusion-exclusion formula) Fix an integer $n \geq 2$ and let $A_1, \ldots, A_n$ be sets. Show that

$$\#\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ I \neq \emptyset}} (-1)^{-1+|I|} \, \#\left(\bigcap_{i \in I} A_i\right).$$

Solution of exercise 9. We show the result by induction. For $n = 1$, there is nothing to do.

Assume that the result is true for a fixed integer $n \geq 1$ and let us show that it is true for $n+1$. Denote by $\widehat{A}_n$ lhe union of $A_1, \ldots, A_n$. Then,

$$\begin{aligned}
\#\left(\widehat{A}_n \cup A_{n+1}\right) &= \#(\widehat{A}_n) + \#A_{n+1} - \#(\widehat{A}_n \cap A_{n+1}) \\
&= \#(\widehat{A}_n) + \#A_{n+1} - \#\left(\bigcup_{i=1}^{n} A_i \cap A_{n+1}\right) \\
&= \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{-1+|I|} \, \#\left(\bigcap_{i \in I} A_i\right) + \#A_{n+1} - \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{-1+|I|} \, \#\left(\bigcap_{i \in I} A_i \cap A_{n+1}\right) \\
&= \sum_{\substack{I \subseteq \{1,\ldots,n+1\} \\ I \neq \emptyset, n+1 \notin I}} (-1)^{-1+|I|} \, \#\left(\bigcap_{i \in I} A_i\right) + \#A_{n+1} + \sum_{\substack{I \subseteq \{1,\ldots,n+1\} \\ I \neq \{n+1\}, n+1 \in I}} (-1)^{-1+|I|} \, \#\left(\bigcap_{i \in I} A_i\right) \\
&= \sum_{\substack{I \subseteq \{1,\ldots,n+1\} \\ I \neq \emptyset}} (-1)^{-1+|I|} \, \#\left(\bigcap_{i \in I} A_i\right),
\end{aligned}$$

□

Exercise 10. Fix an integer $n \geq 1$. A permutation $\{x_1, x_2, \ldots, x_{2n}\}$ of the elements $1, 2, \ldots, 2n$ is a rearrangement of these $2n$ numbers in a different order. It is said to have property T if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$. Show that there are more permutations with property T than without.

For example, for $n = 2$, the permutations which do not have the property T are

$$\{1234, 1432, 2143, 2341, 3214, 3412, 4123, 4321\}$$

and the permutations which have the property T are

$$\{1234, 1324, 1342, 1423, 2134, 2314, 2413, 2431, 3124, 3142, 3241, 4132, 4213, 4231, 4312\}.$$

Hint. If $(x_1, \ldots, x_{2n})$ is a permutation which does not have the property T, you may consider a function f defined by $f((x_1, \ldots, x_{2n})) = (x_2, x_3, \ldots, x_k, x_1, x_{k+1}, \ldots, x_{2n})$ where k is the unique index such that $|x_1 - x_k| = n$. For example, $f(4321) = 3241$.

Solution of exercise 10. Let A be the set of permutations which do not have the property T and let B be the set of permutations $(x_1, \ldots, x_{2n})$ such that $|x_i - x_{i+1}| = n$ for exactly one i in $\{1, 2, \ldots, 2n-1\}$. Then the function f defined in the hint is a well-defined function $f : A \to B$. Indeed, if $(x_1, \ldots, x_{2n})$ is a permutation which does not have the property T, applying f creates only one i such that $|x_i - x_{i+1}| =$

n.

Now, we claim that f is injective. We can either establish this claim by hand, or simply note that if $g : B \to A$ is the function defined by $g((y_1,\dots,y_{2n})) = (y_{k+1}, y_1, \dots, y_k, y_{k+2}, \dots, y_{2n})$ where k is the unique integer such that $|y_k - y_{k+1}| = n$, then $g(f((x_1,\dots,x_{2n}))) = (x_1,\dots,x_{2n})$ for every $(x_1,\dots,x_{2n}) \in A$, which shows that f is injective.

Therefore $\#B \geq \#A$. But permutations of B are permutations which have the property T, and there are more permutations which have the property T than pemutations of B (for example $(1, n+1, 2, n+2, 3, \dots, 2n)$ is such an example, since there are two indices i such that $|x_i - x_{i+1}| = n$).

We conclude that there are more permutations with property T than without. □

4 Fun exercise (optional)

The solution of this exercise will be available on the course webpage at the end of week 8.

Exercise 11. Consider an equilateral triangle with side n, subdivised in small unit triangles as in Fig. 1. A capybara starts from the top triangle and wants to go down. He can only move to adjacent triangles, without going back to a visited triangle and cannot go upwards. He stops when reaching the bottom row. See Figure 1 for an example with $n = 5$. In how many ways can the capybara reach the bottom row when $n = 2017$?

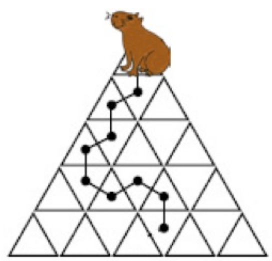

Figure 1: Example of a path reaching the bottom row .

Solution of exercise 11. More generally, let $f(n)$ be the number of such paths.

Label the horizontal line segments in the triangle $\ell_1, \ell_2, \dots$ as in the diagram below. Since the path goes from the top triangle to a triangle in the bottom row and never travels up, the path must cross each of $\ell_1, \ell_2, \dots, \ell_{n-1}$ exactly once. The diagonal lines in the triangle divide ℓ_k into k unit line segments and the path must cross exactly one of these k segments for each k. (In the diagram below, these line segments have been highlighted.) The path is completely determined by the set of $n-1$ line segments which are crossed. So as the path moves from the kth row to the $(k+1)$st row, there are k possible line segments where the path could cross lk. Since there are $1 \cdot 2 \cdots (n-1) = (n-1)!$ ways that the path could cross the $n-1$ horizontal lines, and each one corresponds to a unique path, we get $f(n) = (n-1)!$.

□