Week 9: Binomial coefficients

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1 Important exercises

The solutions of the exercises which have not been solved in some group will be available on the course webpage.

Exercise 1. Give the "simplest" possible expressions (using as less possible characters as possible) for $\binom{10}{9}, \binom{n}{2}, \binom{3}{5}, \binom{2n}{n}$.

Solution of exercise 1. We apply the formula $\binom{n}{k} = \frac{n!}{(n-k)!k!}$:

$$\binom{10}{9} = \frac{10!}{9!1!} = \frac{10 \times 9 \times 8 \times \cdots \times 2 \times 1}{9 \times 8 \times \cdots \times 2 \times 1 \times 1} = 10$$

$$\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}$$

$$\binom{3}{5} = 0 \qquad \text{(by convention)}$$

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n!)^2}.$$

□

Exercise 2. Consider a group of 10 students. We want to select four people for a team project.

1. How many different teams can be chosen?
2. Assume that there are two members of the group who refuse to work together for this project. How many four-person teams can be formed?

Solution of exercise 2.

1. By definition there are $\binom{10}{4} = 210$ distinct possible teams.
2. Let us call these two people A, B. We use the "complement rule" and write

$$\{\text{Teams that do not contain both } A \text{ and } B\} = \{\text{Teams}\} \setminus \{\text{Teams that contain both } A \text{ and } B\}$$

Therefore the solution is $\binom{10}{4} - \binom{8}{2} = 210 - 28 = 182$.

□

Exercise 3.

1. What is the coefficient of a^2b^2 in the expansion of $(a+b)^4$?
2. What are the coefficients of x^{56}, x^{59} and x^{62} in the expansion of $(x^7 + x^{10} + x^{13})^8$?
3. What is the coefficient of $a^2b^5c^3$ in the expansion of $(a+b+c)^{10}$?

Solution of exercise 3.

1. When we expand the quantity $(a+b)^4$ we obtain $aaaa+aaab+aabb+aaba+\cdots$, that is, in other words a sum of 2^4 terms of the form $XXXX$ with each X being a or b. Therefore the coefficient of a^2b^2 is the number of terms $XXXX$ with two a and two b. The number of such terms is given by the number of ways of choosing two positions for the a (the two others are taken then by b), so that the answer is $\binom{4}{2}=\frac{4!}{2!2!}=6$.

   Remark. One can also use the Binomial theorem.

2. First notice that
$$(x^7+x^{10}+x^{13})^8 = x^{56}(1+x^3+x^6)^8.$$
   Therefore the coefficients of x^{56}, x^{59} and x^{62} in the expansion of $(x^7+x^{10}+x^{13})^8$ are respectively the coefficients of 1, x^3 and x^6 in the expansion of $(1+x^3+x^6)^8$.

   - The coefficient of 1 is 1. Indeed, in order to get 1 in the expansion of $(1+x^3+x^6)^8$, there is only one way: one has to choose 1 in each factor.
   - The coefficient of x^3 is 8. Indeed, in order to get x^3 in the expansion of $(1+x^3+x^6)^8$, one has to choose one factor where one keeps x^3, and one keeps 1 in all the others. There are $8=\binom{8}{1}$ ways to do this.
   - The coefficient of x^6 is 36. Indeed, since $6=6+0+0+0+0+0=3+3+0+0+0+0$, there are several ways to get x^6 in the expansion $(1+x^3+x^6)^8$: either choose one factor where one keeps x^6, and one keeps 1 in all the others (there are $8=\binom{8}{1}$ ways to do this), or one chooses two factors where one keeps x^3, and one keeps 0 in all the others (there are $\binom{8}{2}=28$ ways to this this). In total, we get $36=8+28$.

   Remark. These computations show that
$$(x^7+x^{10}+x^{13})^8 = x^{56}(1+8x^3+36x^6+x^6P(x)),$$
   where P is a polynomial.

3. We have to count the number of distinct terms like $aabbbbbccc, cabbabbccb, \ldots$ (sequences of length 10 with 2 a's, 5 b's and 3 c's). These terms are in bijection with the set of pairs
$$(\text{Subset of } \{1,2,\ldots,10\} \text{ of size } 2, \text{Subset of } \{1,2,\ldots,8\} \text{ of size } 5).$$
   Indeed, we first fix locations of 2 a's (which amounts to choosing a subset of size 2 of $\{1,2,\ldots,10\}$), then we have to choose the locations of 5 b's among the 8 remaining locations (once this is done, this fixes the locations of the 3 remaining c's). The answer is thus
$$\binom{10}{2}\times\binom{8}{5}=\frac{10!}{2!5!3!}=2520.$$

(Note that if we first choose, say, the locations of c's and then the locations of a's we obtain of course the same result:

$$\binom{10}{3} \times \binom{7}{2} = \frac{10!}{2!5!3!} = 2520.)$$

□

Exercise 4. Let $n \geq 6$ be an integer. In how many ways is it possible to place n different balls into n numbered bins in such a way that there are exactly 3 balls in the first bin and exactly 2 balls in the second bin? Give the simplest possible formula.

Solution of exercise 4. This amounts to choosing the 3 balls that go into the first bin ($\binom{n}{3}$ choices), then the 2 balls among the $n-3$ remaining balls that fall into the second bin ($\binom{n-3}{2}$ choices) and finally the number of ways to place the $n-5$ balls in the last $n-2$ bins ($(n-2)^{n-5}$ choices, since for every ball we have $n-2$ choices). The total number of ways is therefore:

$$\binom{n}{3}\binom{n-3}{2}(n-2)^{n-5} = \frac{n!}{(n-3)!3!}\frac{(n-3)!}{(n-5)!2!} \cdot (n-2)^{n-5} = \frac{n!}{12(n-5)!} \cdot (n-2)^{n-5},$$

which is equal to

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{12} \cdot (n-2)^{n-5}.$$

□

2 Homework exercise

There are no homework exercises this time ☹.

3 More involved exercises (optional)

The solution of these exercises will be available on the course webpage at the end of week 9.

Exercise 5. Fix integers $k, n \geq 1$. The goal of this exercise is to show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

1. Give an "algebraic" proof.
2. Give a "combinatorial" proof by counting in two different ways the number of subsets of cardinality k of $\{1, 2, \dots, n\}$ having a distinguished element (a "chieftess").

Solution of exercise 5.

1. We have

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$$

and

$$n\binom{n-1}{k-1} = n\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = \frac{n!}{(k-1)!(n-k)!}.$$

These two quantities are indeed equal.

2. *First way of counting:* first choose a subset of cardinality k ($\binom{n}{k}$ ways) and then choose a distinguished element (k ways). By the multiplicative principle, this gives $k\binom{n}{k}$ ways.

   Second way of counting: first choose a chieftess (n ways), and then add the $k-1$ other elements ($\binom{n-1}{k-1}$ ways). By the multiplicative principle, this gives $n\binom{n-1}{k-1}$ ways.

   We conclude that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

□

Exercise 6. 1. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, recall the expansion of $(1+x)^n$ as a sum. Deduce simple formulas for $\sum_{k=0}^{n} \binom{n}{k}$ and $\sum_{k=0}^{n} k\binom{n}{k}$.

2. For $n \geq 1$ simplify $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n}$.

3. Given a set X of cardinality n, show that the number of subsets of X with odd cardinality is equal to the number of subsets of X with even cardinality.

Solution of exercise 6.

1. We recall the formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n. \qquad (\star)$$

   If we put $x = 1$ in the above formula we obtain

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k = \sum_{k=0}^{n} \binom{n}{k}.$$

   If we take the derivative with respect to x of both sides in ($\star$) we obtain

$$((1+x)^n)' = \left(\sum_{k=0}^{n} \binom{n}{k} x^k \right)'$$
$$n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}.$$

   If we now put $x = 1$ we get

$$n2^{n-1} = \sum_{k=0}^{n} \binom{n}{k} \times k.$$

2. If we put $x = -1$ in ($\star$) we obtain

$$(1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \binom{n}{0}(-1)^0 + \binom{n}{1}(-1)^1 + \binom{n}{2}(-1)^2 + \cdots + \binom{n}{n}(-1)^n.$$

which simplifies into

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n}.$$

3. By the previous question,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots.$$

The left-hand side of the equality counts the number of subsets of X with even cardinality and the right-hand side of the equality counts the number of subsets of X with even cardinality, which entails the desired result.

$\square$

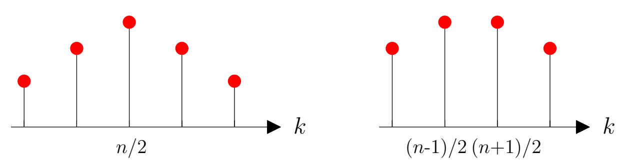
Exercise 7. Let $n \geq 1$ be fixed. What is the largest of the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots$?
(Hint: Put $f(k) = \binom{n}{k}$ and compute $f(k+1)/f(k)$.)

Solution of exercise 7. For $k = 0, \ldots, n-1$, we compute

$$\frac{f(k+1)}{f(k)} := \frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{\frac{n!}{(k+1)!(n-k-1)!}}{\frac{n!}{k!(n-k)!}} = \frac{n-k}{k+1}.$$

Thus we have

$$\frac{f(k+1)}{f(k)} \begin{cases} \geq 1 & \text{for } n-k \geq k+1 \textit{ i.e. } k \leq (n-1)/2. \\ < 1 & \text{for } n-k < k+1 \textit{ i.e. } k > (n-1)/2. \end{cases}$$



Finally,

- If n is odd, $\max_k \binom{n}{k} = \binom{n}{(n-1)/2} = \binom{n}{(n+1)/2}$.
- If n is even, $\max_k \binom{n}{k} = \binom{n}{n/2}$.

$\square$

Exercise 8. For an integer $n \geq 1$, consider the identity ($\star$):

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

1. Check that ($\star$) is true for $n = 2$.

2. Consider a group of $2n$ students with n boys and n girls. By counting in two different manners the number of ways to choose group of n students, find a combinatorial proof of ($\star$).

3. Deduce that $\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$.

Solution of exercise 8.

1. We have

$$\sum_{k=0}^{2}\binom{2}{k}\binom{2}{2-k}=\binom{2}{0}\binom{2}{2}+\binom{2}{1}\binom{2}{1}+\binom{2}{2}\binom{2}{0}=1+2\times 2+1=6,$$

which is indeed equal to $\binom{4}{2}$.

2. *First way of counting:* choose the n students among the $2n$ students. This gives $\binom{2n}{n}$ ways.

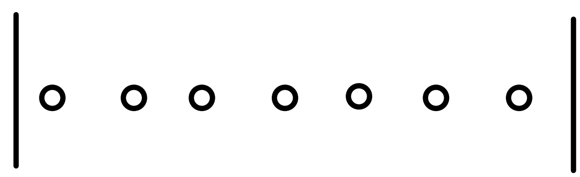
Second way of counting: For a given $0\le k\le n$, choose k girls and $n-k$ boys, which can be done in $\binom{n}{k}\binom{n}{n-k}$ ways. Since the set of all choices can be writen as a disjoint union $E_0\cup E_1\cup\cdots\cup E_n$ where E_k represents the choices with k girls. By the "addition principle", this gives $\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$ ways.

3. This follows from the fact that $\binom{n}{k}=\binom{n}{n-k}$ for every $0\le k\le n$.

□

Exercise 9. In how many ways is it possible to put 5 identical coins into 3 different pockets?

Solution of exercise 9. Consider the following figure (with two vertical lines):



Replace each circle by either a coin, or a vertical line so that there is in total 4 vertical lines and 5 coins. The coins between the first two vertical lines will be those in the first pocket, the coins between the second and third vertical lines will be those in the second pocket and the coins after the third vertical line will be those in the third pocket.

Therefore, putting 5 identical coins in 3 different pockets amounts to choosing the position of two new vertical lines among the 7 possible slots. The answer is therefore $\binom{7}{2}=21$.

More generally, the same argument shows that there are $\binom{a+b-1}{b-1}=\binom{a+b-1}{a}$ ways to place a identical coins into b different pockets, see the next exercise. □

Exercise 10. Fix integers $1\le m\le n$.

1) In how many ways can we choose a sequence $(x_1,x_2,\ldots,x_m)$ of nonnegative integers such that $x_1+x_2+\cdots+x_m=n$?

2) In how many ways can we choose a sequence $(x_1,x_2,\ldots,x_m)$ of (strictly) positive integers such that $x_1+x_2+\cdots+x_m=n$?

Solution of exercise 10.

1) The idea is to adapt the solution of the previous exercise. Consider the same figure as in the solution of the previous exercise, with $m+n-1$ circles. Replace each circle by either a coin, or a vertical line so that there is in total $m-1$ vertical lines and n coins. Letting x_1 be the number of coins between the first two vertical lines, x_2 be the coins between the second and third vertical lines the coins and so on shows that the number of ways to choose a sequence $(x_1,x_2,\ldots,x_m)$ of nonnegative integers such that $x_1+x_2+\cdots+x_m = n$ is equal to number of ways of choosing n circles among $m+n-1$, which is $\binom{n+m-1}{n}$.

2) By considering $(x_1-1, x_2-1, \ldots, x_m-1)$ instead of $(x_1,x_2,\ldots,x_m)$, one sees that choosing a sequence $(y_1,y_2,\ldots,y_m)$ of nonnegative integers such that $y_1+y_2+\cdots+y_m = n-m$ is equivalent. We saw in the previous question that this number is $\binom{n-1}{m-1}$.

Remark. More formally, if $E = \{(x_1,x_2,\ldots,x_m) \in \{1,2,3,\ldots\}^m : x_1+x_2+\cdots+x_m = n\}$ and $F = \{(y_1,y_2,\ldots,y_m) \in \{0,1,2,\ldots\}^m : y_1+y_2+\cdots+y_m = n-m\}$, the function

$$\begin{array}{cccc} f: & E & \to & F \\ & (x_1,x_2,\ldots,x_m) & \mapsto & (x_1-1,x_2-1,\ldots,x_m-1) \end{array}$$

is a bijection.

It is also possible to solve this question without using the first one by defining $G = \{A \subseteq \{1,2,\ldots,n-1\} : \#A = m-1\}$, the function

$$\begin{array}{cccc} g: & E & \to & G \\ & (x_1,x_2,\ldots,x_m) & \mapsto & \{x_1, x_1+x_2, x_1+\cdots+x_{m-1}\} \end{array}$$

and checking that g is a bijection. □

Exercise 11. Given integers $0 \leq m \leq n$, show that $\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m}$.

Solution of exercise 11. First notice that only indices k such that $k \geq m$ give a positive contribution to the sum.

We give a combinatorial proof by counting in two different manners the number N of subsets A, B of $\{1,2,\ldots,n\}$ such that $A \subseteq B$ and $\#A = m$.

First, to build such A and B, we can first choose A of cardinality m ($\binom{n}{m}$ choices), and then to obtain B add to A a subset of $\{1,2,\ldots,n\}\backslash A$ having $n-m$ elements so that B has $m+(n-m) = n$ elements (2^{n-m} choices). Therefore, $N = \binom{n}{m}2^{n-m}$.

Second, to build such A and B one may first choose B with any cardinality k with $m \leq k \leq n$ (if B has cardinality k, $\binom{n}{k}$ choices), then choose A with cardinality m as a subset of B ($\binom{k}{m}$ choices if B has cardinality k). Therefore $N = \sum_{k=0}^{n}\binom{n}{k}\binom{k}{m}$. □

Exercise 12. The game of poker is played with a deck of 52 cards: Deck $= \{1,\ldots,10,J,Q,K\} \times \{\heartsuit,\clubsuit,\diamondsuit,\spadesuit\}$. A *hand* is a subset of five cards. We consider the following particular hands:

Royal Flush. $10, J, Q, K, 1$ of the same symbol.

Full house. 3 cards of one value, 2 cards of another value. Example : $\{K\heartsuit, 9\clubsuit, 9\heartsuit, K\diamondsuit, 9\spadesuit\}$.
Four of a kind. 4 cards of one value, the fifth card can be any other card. Example : $\{K\heartsuit, 9\clubsuit, 9\heartsuit, 9\diamondsuit, 9\spadesuit\}$.

1. How many different hands of five cards are there?
2. How many hands correspond to a Royal flush?
3. How many hands correspond to a Four of a kind?
4. How many hands correspond to a Full house?

Solution of exercise 12.

1. $\binom{52}{5} = 2598960$.
2. Four (obvious).
3. Let $\mathcal{F}$ be the set of hands corresponding to a four-of-a-kind , for example $F = \{K\heartsuit, 9\clubsuit, 9\heartsuit, 9\diamondsuit, 9\spadesuit\} \in \mathcal{F}$. We define the function ϕ by

$$\begin{array}{rccc} \phi: & \mathcal{F} & \to & \{1,2,\ldots,13\} \times \text{Deck} \\ & F & \mapsto & (\text{Value of the four identical cards}, \text{One of the remaining card}). \end{array}$$

   The function ϕ is a bijection from $\mathcal{F}$ to the set of couples (s, C) in $\{1,2,\ldots,13\} \times$ Deck such that C isn't one of the 4 cards taken in s. We obtain

$$\text{card}(\mathcal{F}) = 13 \times (52 - 4) = 624.$$

4. Let $\mathcal{H}$ be the set of hands corresponding to a Full house, for example $H = \{K\heartsuit, 9\clubsuit, 9\heartsuit, K\diamondsuit, 9\spadesuit\} \in \mathcal{H}$. For each $H \in \mathcal{H}$ we define $\phi(H)$ by the triple

$$\phi(H) = ((\text{Value of the triple}, \text{Value of the pair}), \text{Symbols of the triple}, \text{Symbols of the pair}).$$

   (Observe that the two values must be different.) For example,

$$\phi(\{K\heartsuit, 9\clubsuit, 9\heartsuit, K\diamondsuit, 9\spadesuit\}) = ((9, K), \{\clubsuit, \heartsuit, \spadesuit\}, \{\heartsuit, \diamondsuit\}).$$

   The codomain of ϕ is

$$(\text{couples of distinct symbols}) \times (\text{subsets of } \{\heartsuit, \clubsuit, \diamondsuit, \spadesuit\} \text{ of size } 3) \times (\text{subsets of } \{\heartsuit, \clubsuit, \diamondsuit, \spadesuit\} \text{ of size } 2)$$

   The function ϕ is a bijection: for each triple ((1st value, 2d value), 3 Symbols, 2 Symbols) we can recover the unique corresponding hand. We obtain

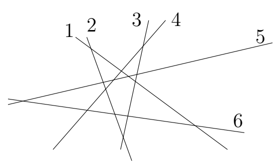
$$\begin{aligned}\mathrm{card}(\mathcal{H}) &= \mathrm{card}\,(\text{pairs of distinct symbols}) \times \mathrm{card}\,(\text{subsets of } \{\heartsuit,\clubsuit,\diamondsuit,\spadesuit\} \text{ of size } 3)\\ &\quad \times \mathrm{card}\,(\text{subsets of } \{\heartsuit,\clubsuit,\diamondsuit,\spadesuit\} \text{ of size } 2)\\ &= 13 \times 12 \times \binom{4}{3} \times \binom{4}{2}\\ &= 3744.\end{aligned}$$

□

4 Fun exercise (optional)

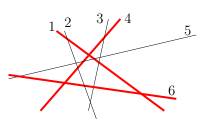
The solution of this exercise will be available on the course webpage at the end of week 9.

Exercise 13. These 6 infinite straight lines are assumed to be in *general position*: three lines never intersect at the same point, and two lines are never parallel.

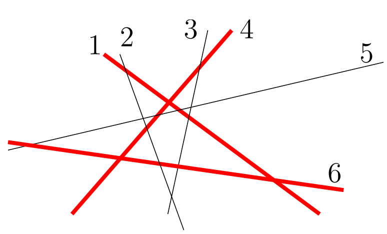


How many triangles are delimited by these lines ? And what if there are n infinite straight lines in general position ?

For example, the lines 1, 4 and 6 delimate one triangle, which is in bold in the following figure:



Solution of exercise 13. To each triangle corresponds a subset of $\{1,2,\ldots,6\}$ of cardinality three, for example this triangle:



corresponds to the subset $\{1,4,6\}$. Conversely, each subset of cardinality three defines a triangle (here we use the fact that lines are in general position).

Hence, there are $\binom{6}{3} = 20$ triangles. More generally, for n lines, there are $\binom{n}{3}$ triangles.

□