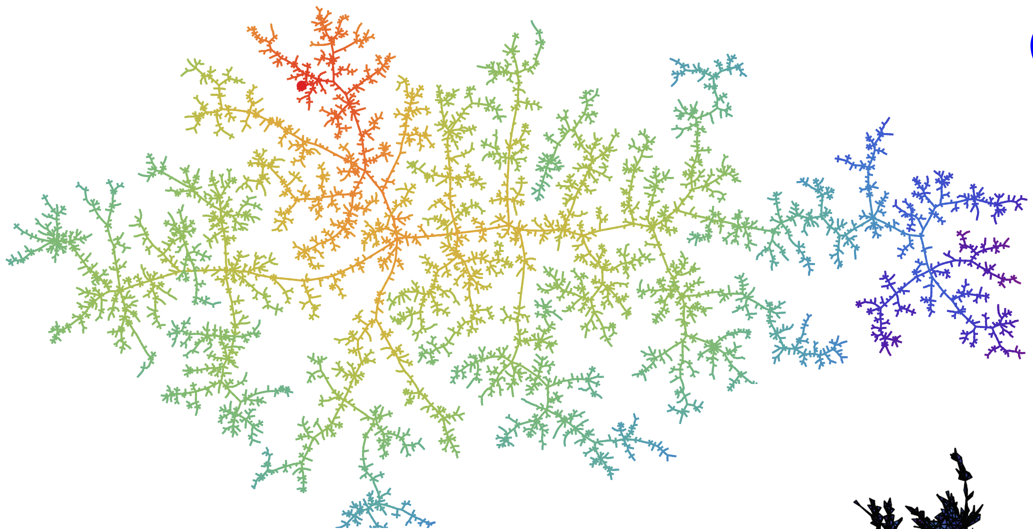
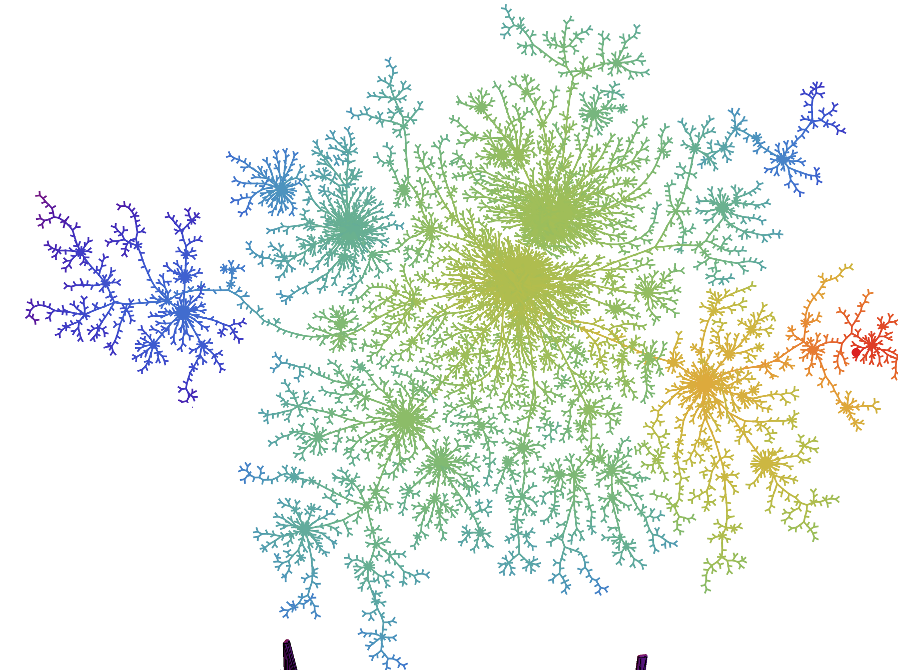
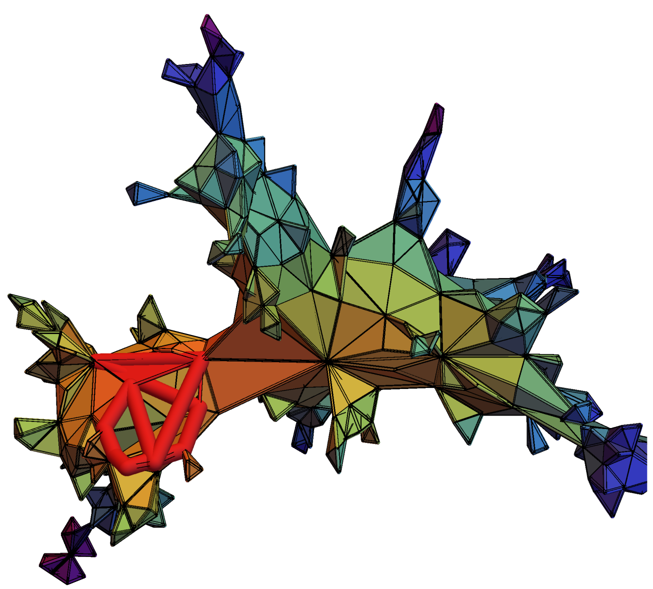
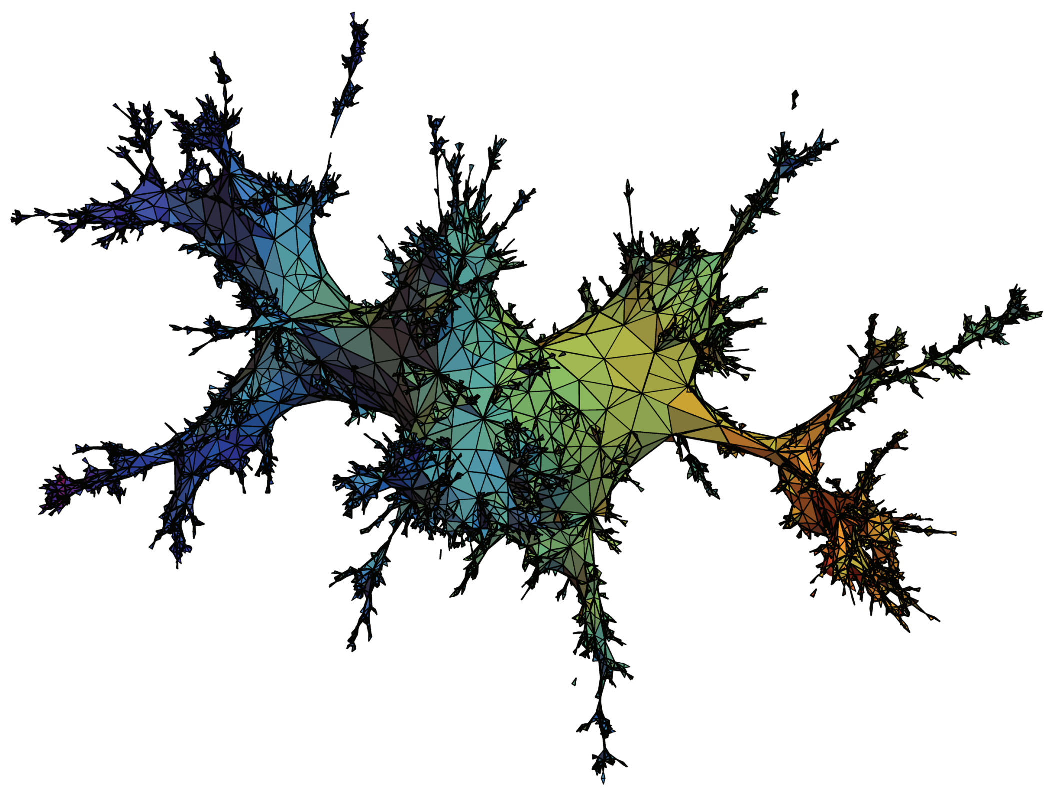
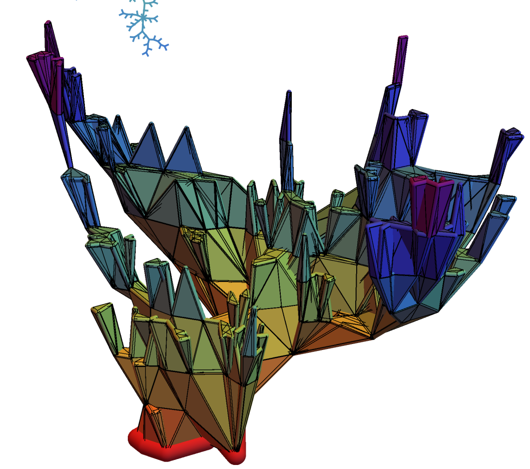
Self-similar growth-fragmentations & random planar maps











Igor Kortchemski (joint work with J. Bertoin, T. Budd, N. Curien)

CNRS & École polytechnique

Stable Processes – November 2016 – Oaxaca

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$\leadsto$ Goal: study a random surface by studying its level sets.

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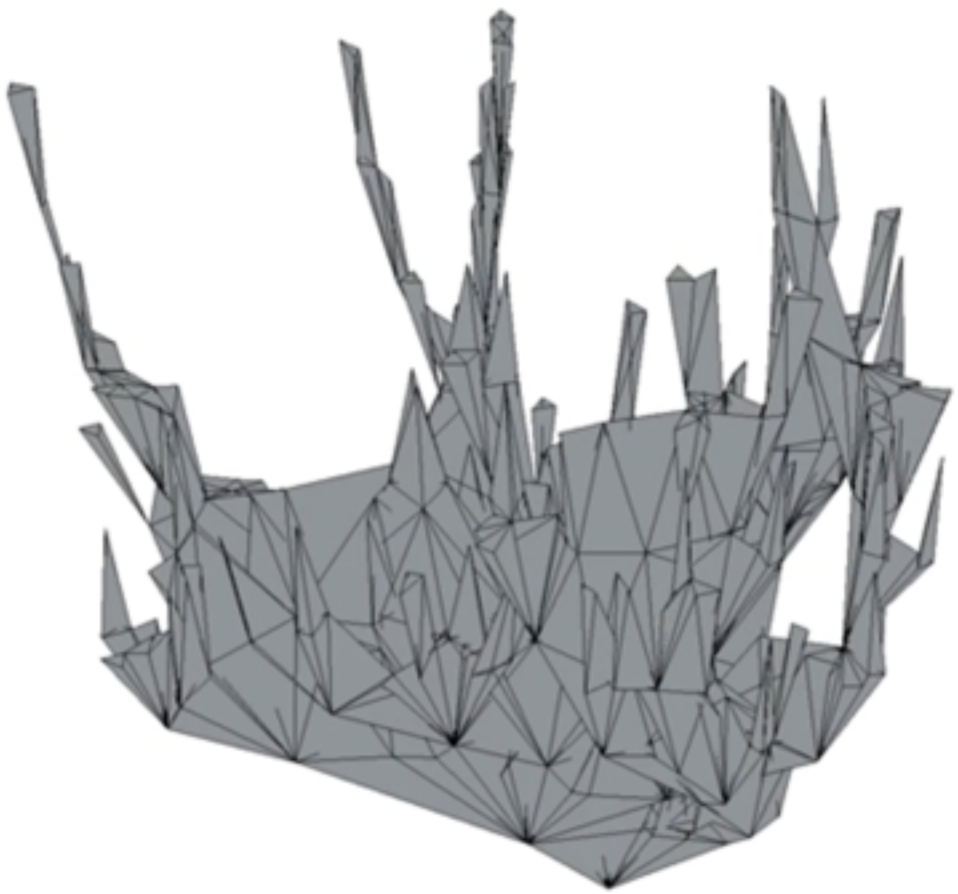

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Approach from the discrete side.

Random surface

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Level sets of
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$$\text{Discretized random surface} \quad \xrightarrow[\text{scaling limit}]{} \quad \text{Random surface}$$

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$\leadsto$ **Convergence in distribution in a certain metric space.**

Outline

I. Planar maps

II. Bienaymé–Galton–Watson trees

III. Random maps and growth-fragmentations

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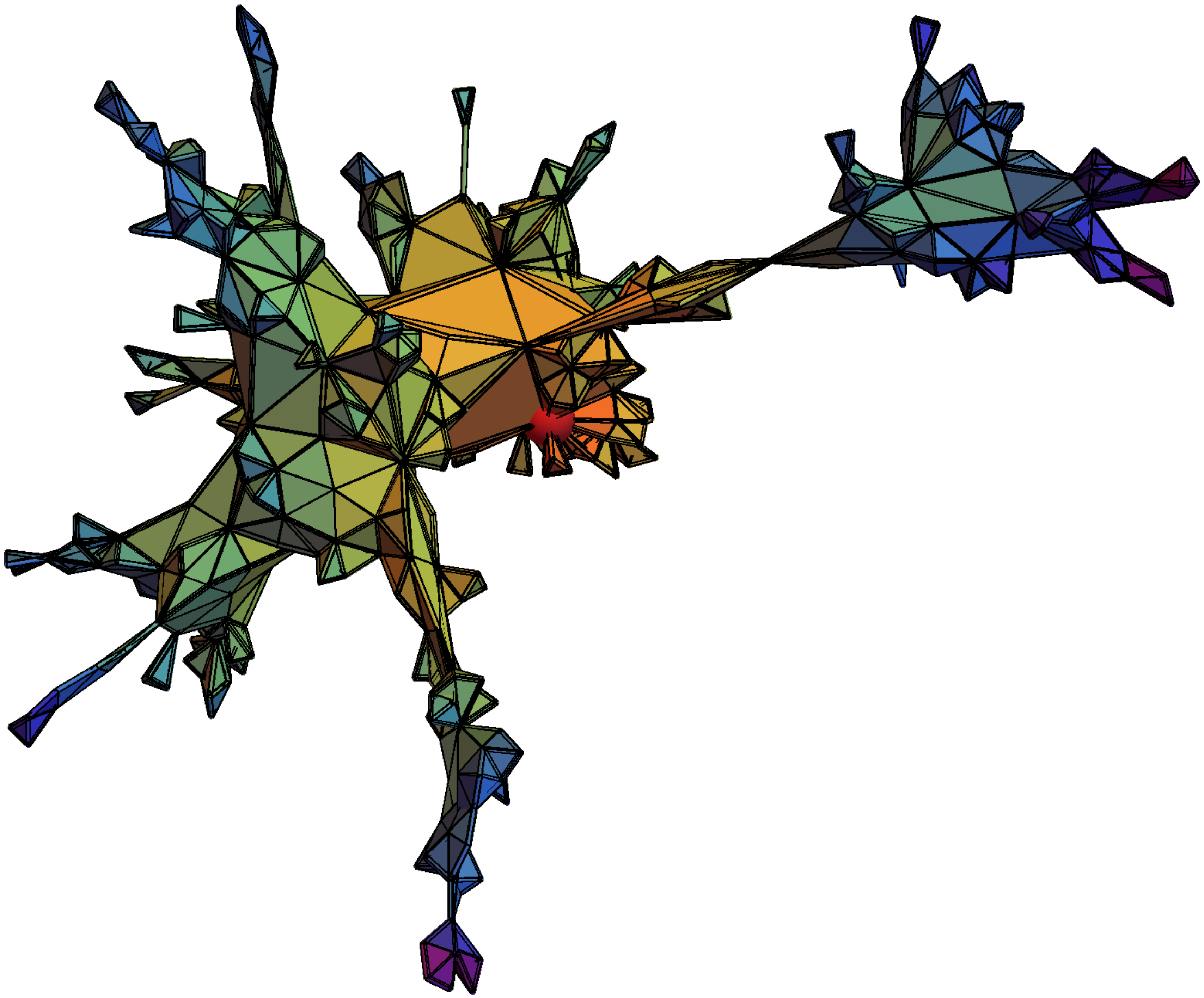

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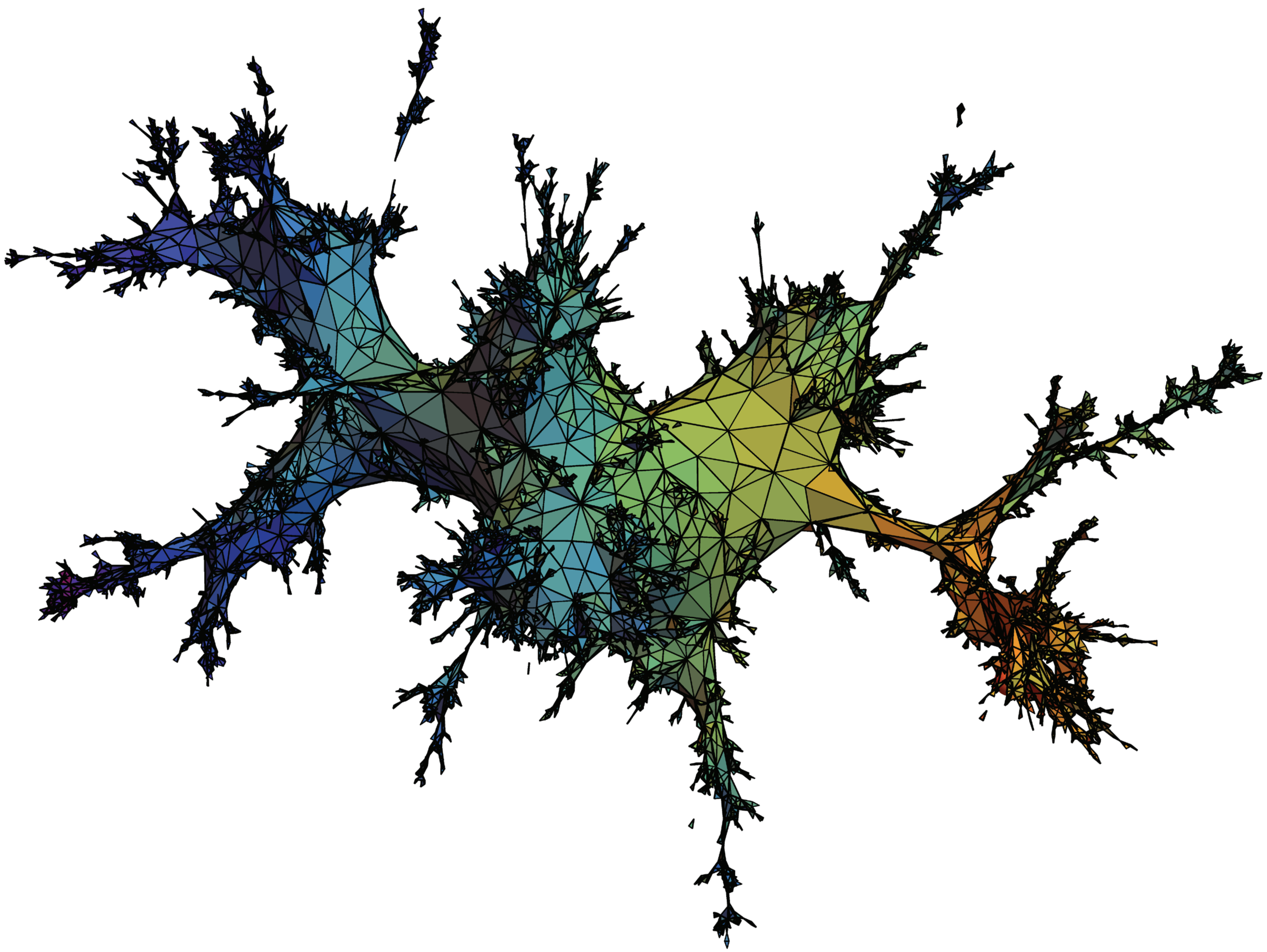

Figure: A large random triangulation

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$\rightsquigarrow$ Other motivations:

– connections with 2D Liouville Quantum Gravity (David, Duplantier, Garban, Kupianen, Maillard, Miller, Rhodes, Sheffield, Vargas, Zeitouni).

– study of random planar maps decorated with statistical physics models (Angel, Berestycki, Borot, Bouttier, Guitter, Chen, Curien, Gwynne, K., Kassel, Laslier, Mao, Ray, Richier, Sheffield, Sun, Wilson).

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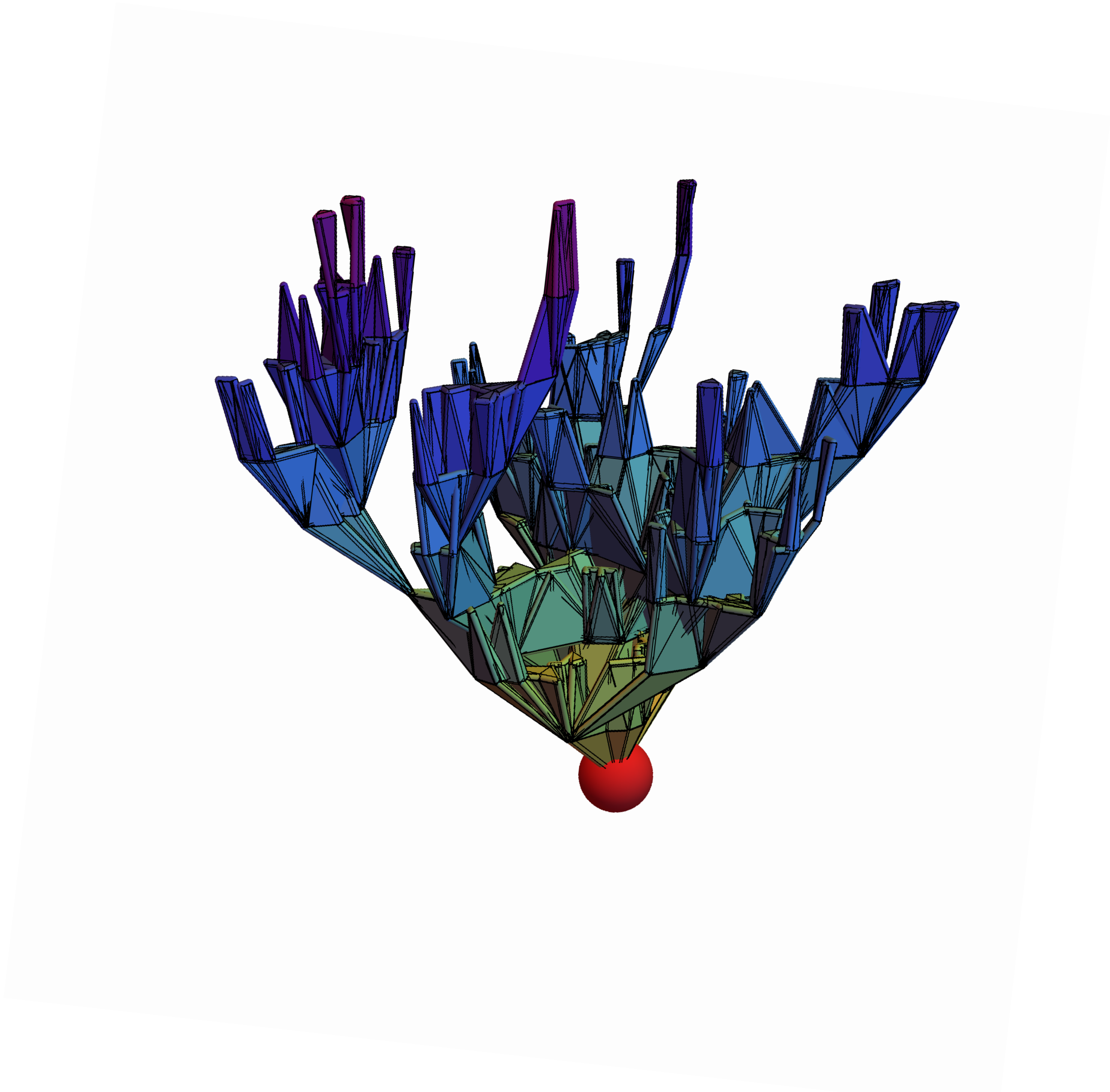

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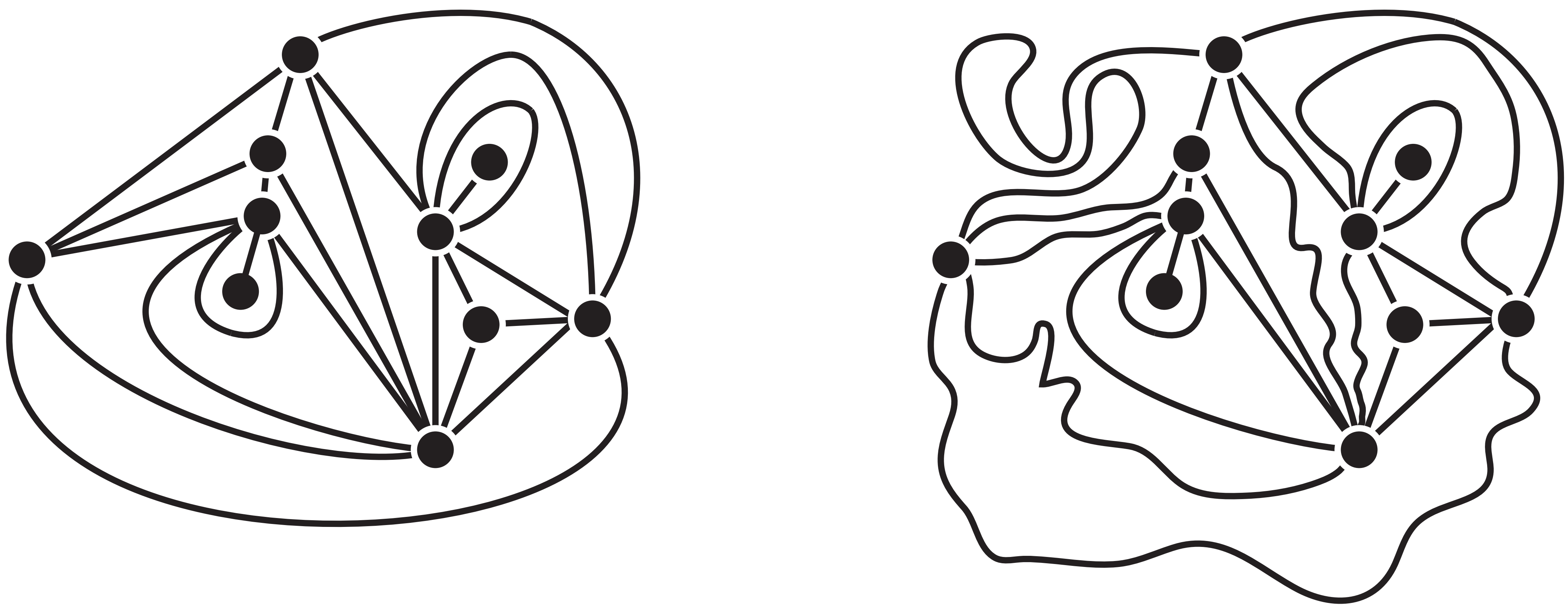

Figure: Two identical maps.

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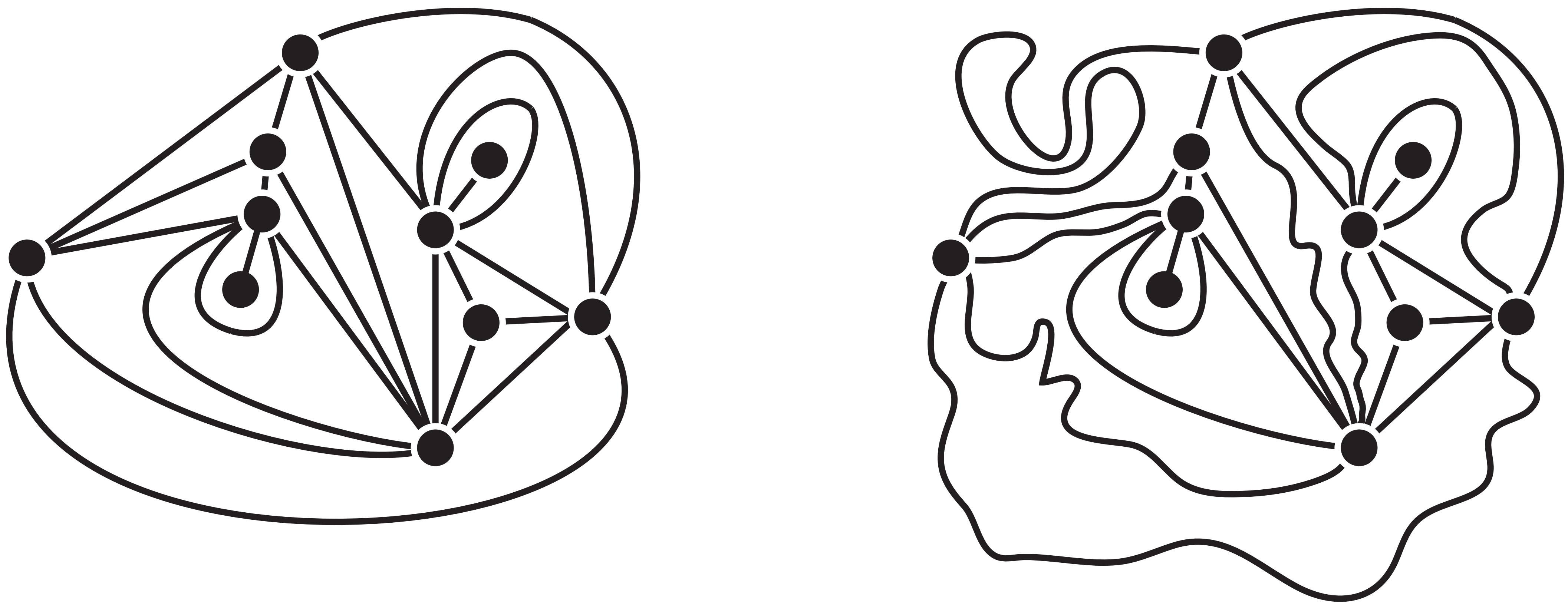

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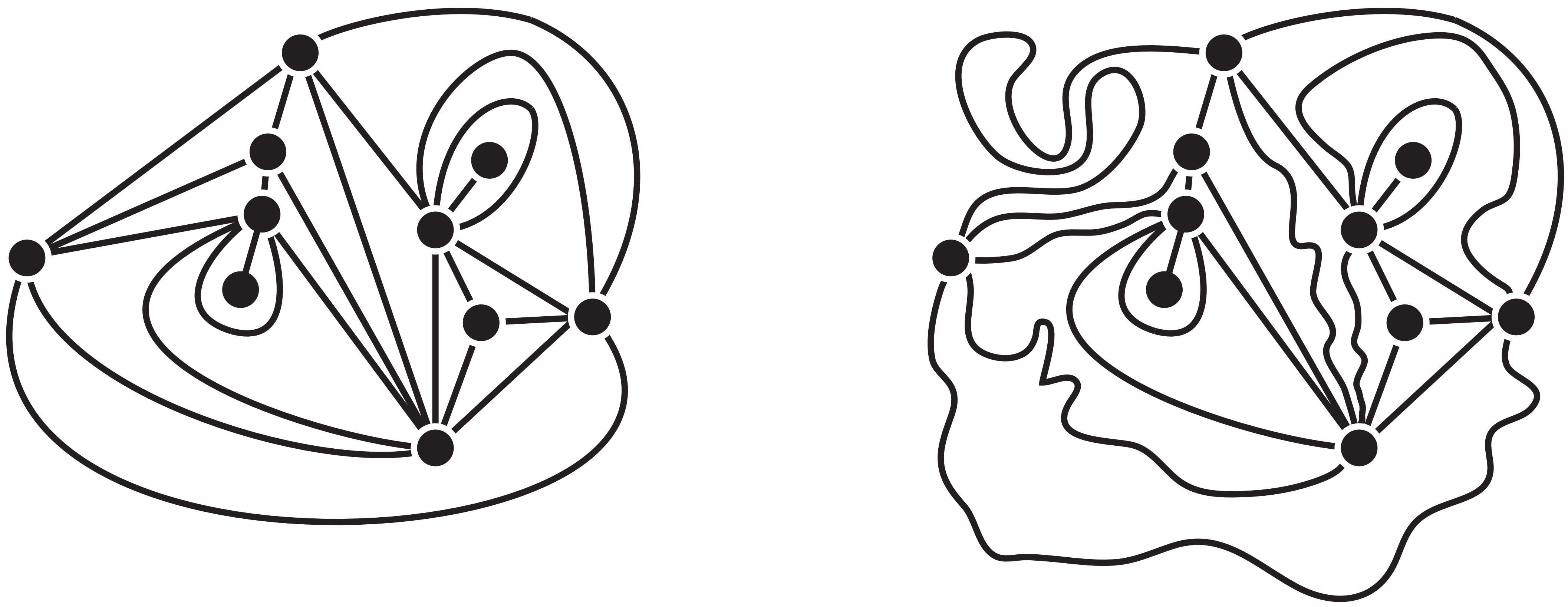

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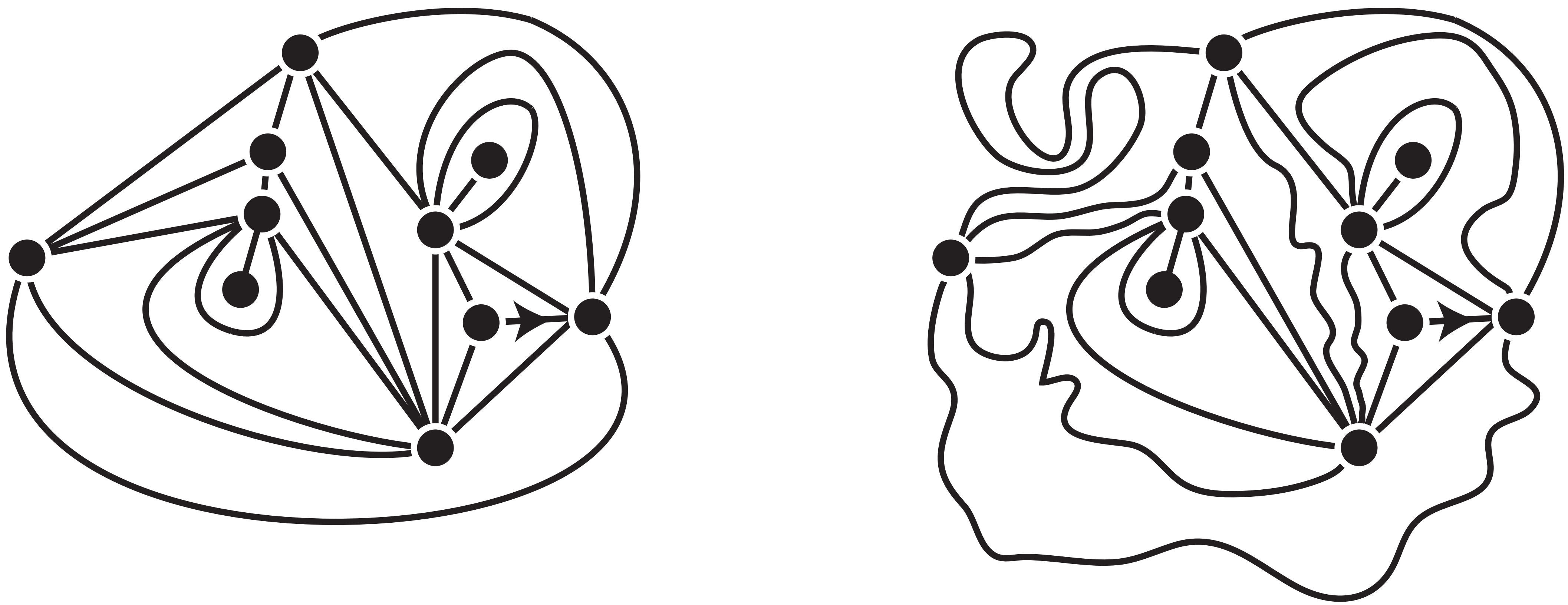

Figure: Two identical rooted triangulations.

Triangulations with a boundary

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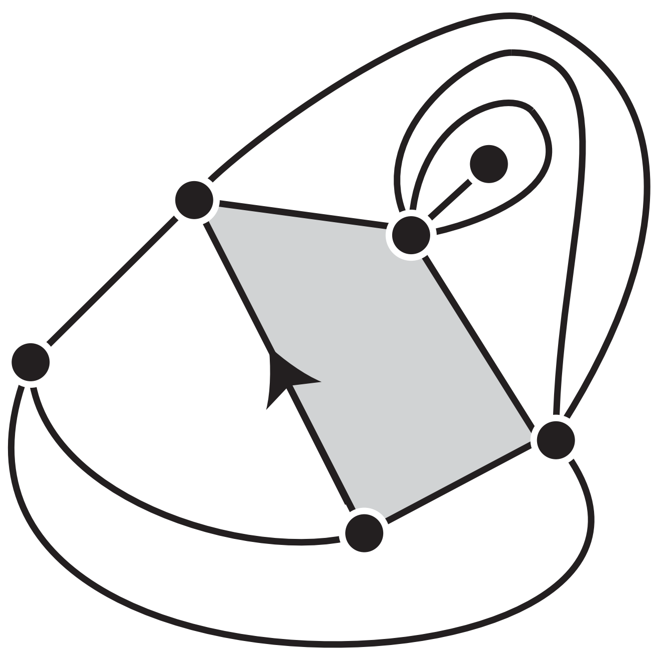

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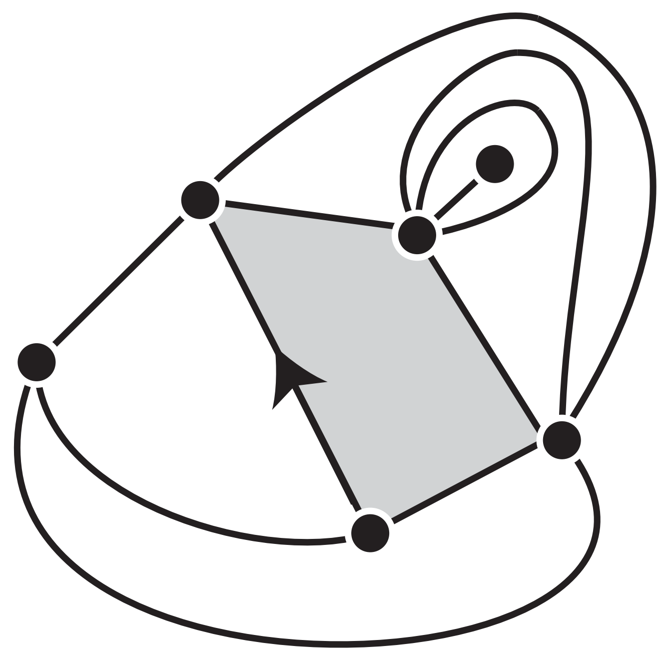

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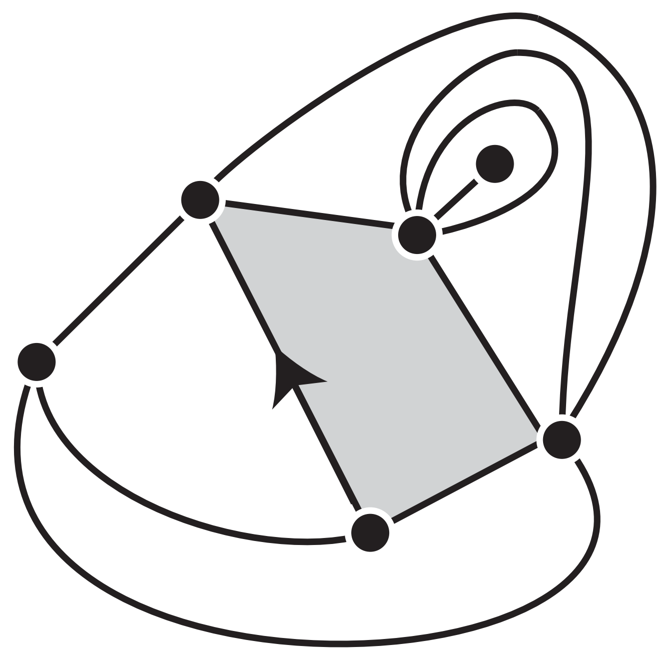

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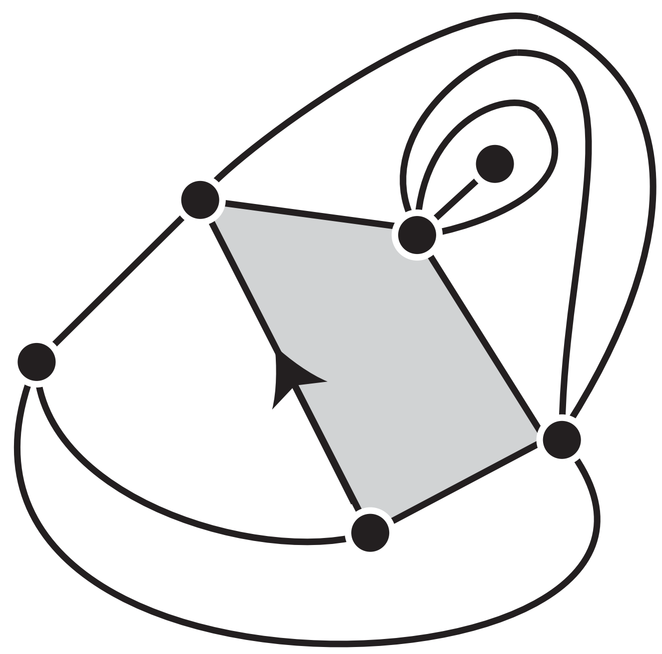

Figure: A triangulation of the 4-gon with two internal vertices (not adjacent to the external face).

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$\leadsto$ what probability measure of planar maps?

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III. Random maps and growth-fragmentations

Plane trees

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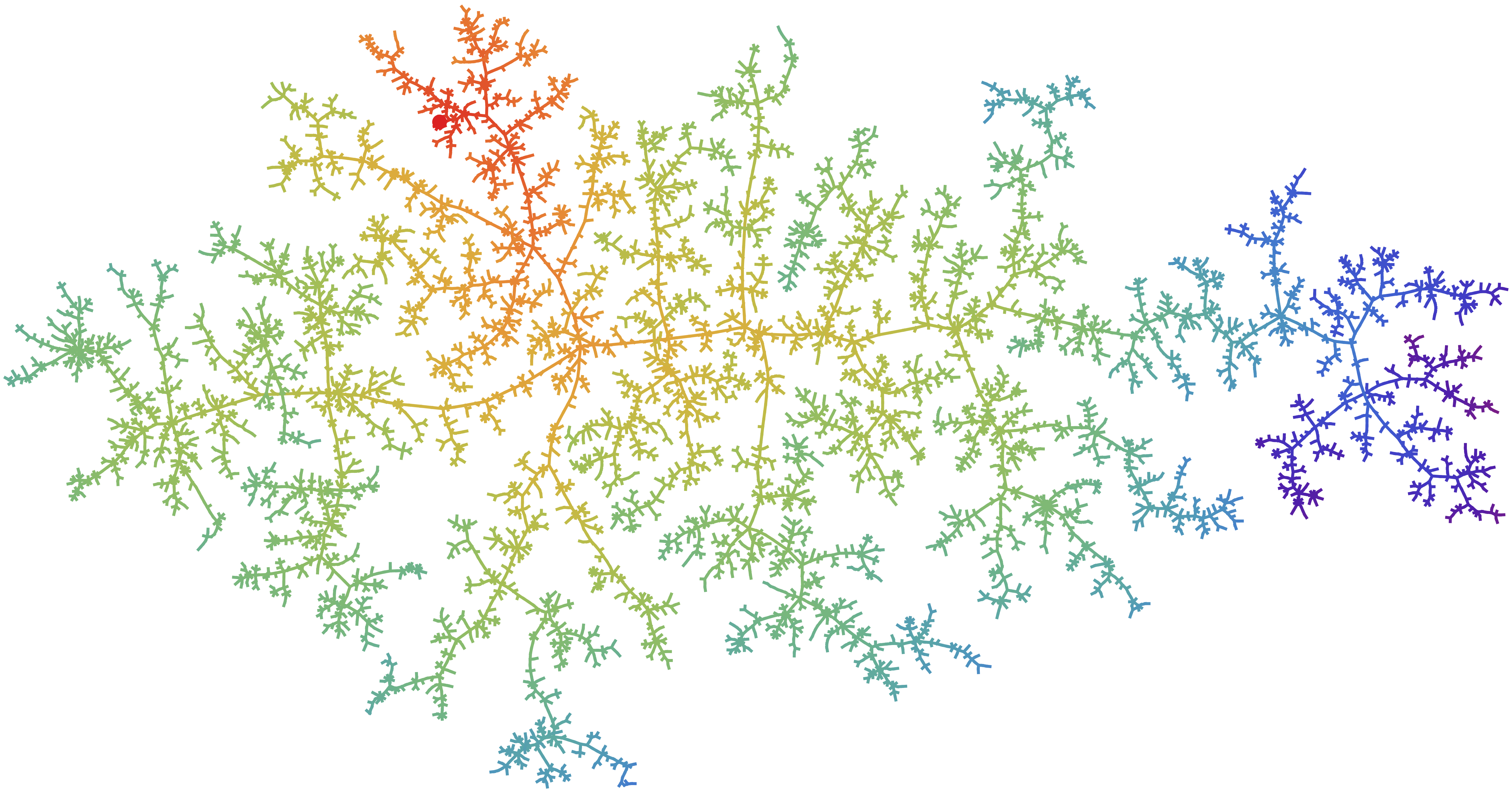
⇝ Natural question: what does a large "typical" plane rooted tree look like?

⇝ Let t_n be a large random plane tree, chosen uniformly at random among all rooted plane trees with n vertices.

A simulation of a large random tree



Uniform plane trees

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The radius of convergence is $1/4$, and by taking $x = 1/4$, one gets a BGW tree with offspring distribution μ.

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Then, if $\mathbb{T}_n$ is the set of all trees with n vertices, for every $\tau \in \mathbb{T}_n$, set

$$\mathbb{P}_n^w(\tau) = \frac{\Omega^w(\tau)}{\sum_{T \in \mathbb{T}_n} \Omega^w(T)}.$$

Scaling limits of large simply generated trees

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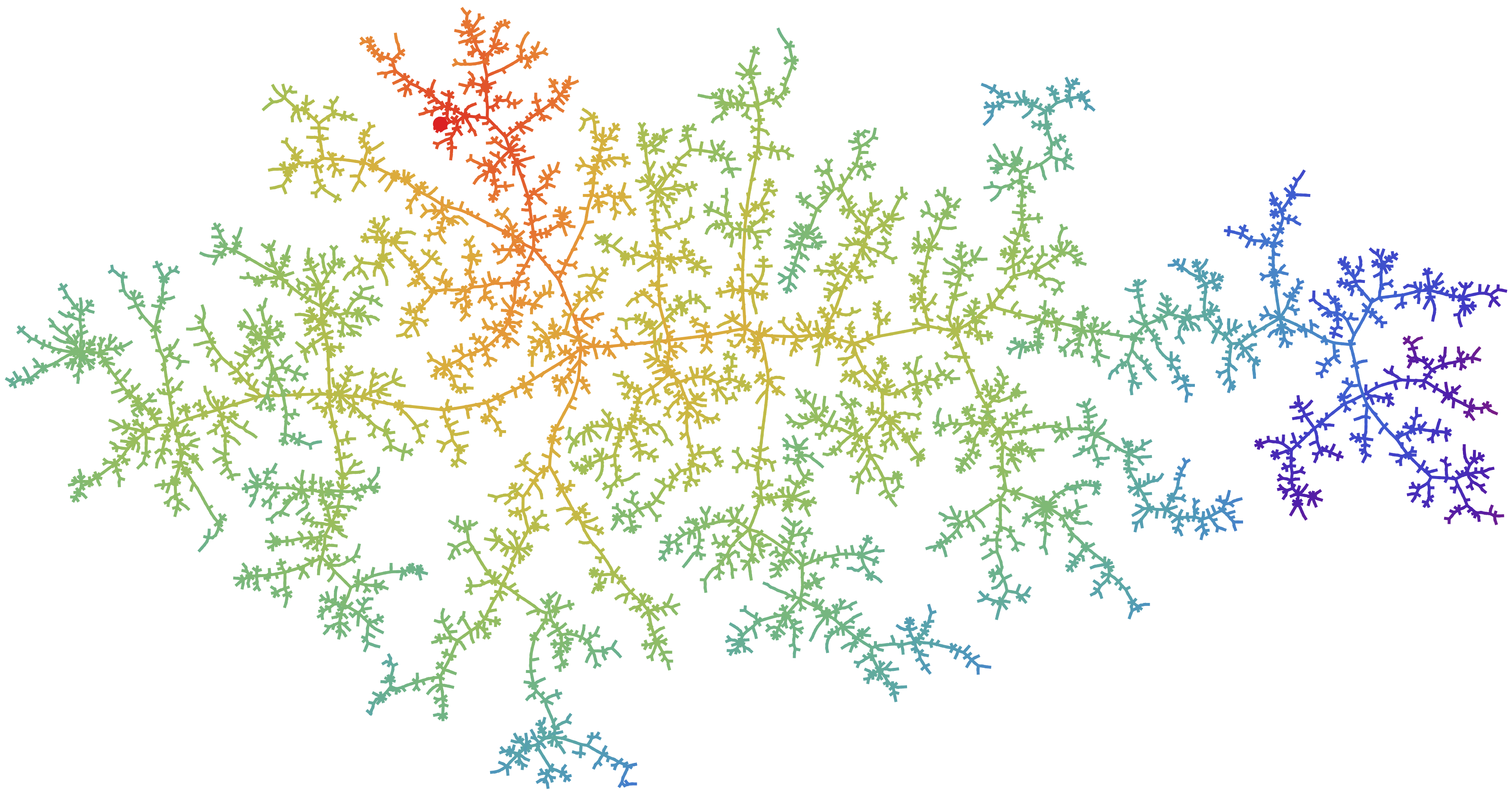

Figure: A non isometric embedding of a realization of the Brownian tree.

Large simply generated trees

↝ If the weight sequence is sufficiently regular, the scaling limit of simply generated trees is the Brownian tree (Aldous).

↝ If the weight sequence has a heavy tail behavior, the scaling limit of simply generated trees is a stable tree (Duquesne, Le Gall, Le Jan).

Large simply generated trees

⇝ If the weight sequence is sufficiently regular, the scaling limit of simply generated trees is the Brownian tree (Aldous).

⇝ If the weight sequence has a heavy tail behavior, the scaling limit of simply generated trees is a stable tree (Duquesne, Le Gall, Le Jan).

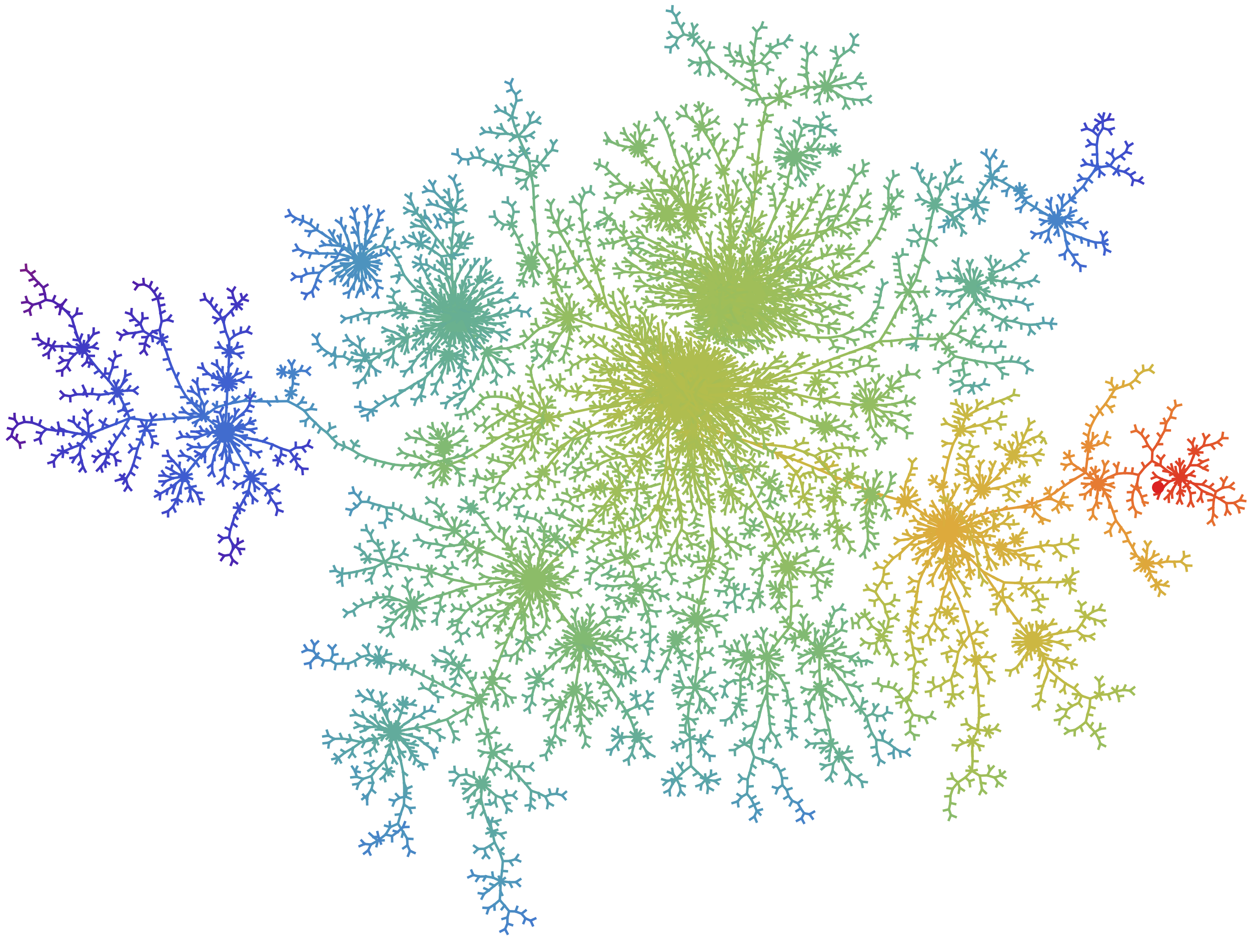

Figure: A non isometric embedding of a realization of a stable tree with index 1.2.

Outline

I. Planar maps

II. Bienaymé–Galton–Watson trees

III. Scaling limits of level sets of random maps

Random maps

$\rightsquigarrow$ What probability distribution on plane triangulations?

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For BGW trees: how to force a BGW tree to be large? One way is to condition it to have size p. Another way is to consider a forest of p BGW trees.

⇝ Similarly, for planar triangulations we will take a Boltzmann distribution on planar triangulations with a large boundary p.

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A triangulation of the p-gon chosen at random with probability

$$(12\sqrt{3})^{-\#(\text{internal vertices})} Z(p)^{-1}$$

is called a **Boltzmann triangulation of the p-gon.**

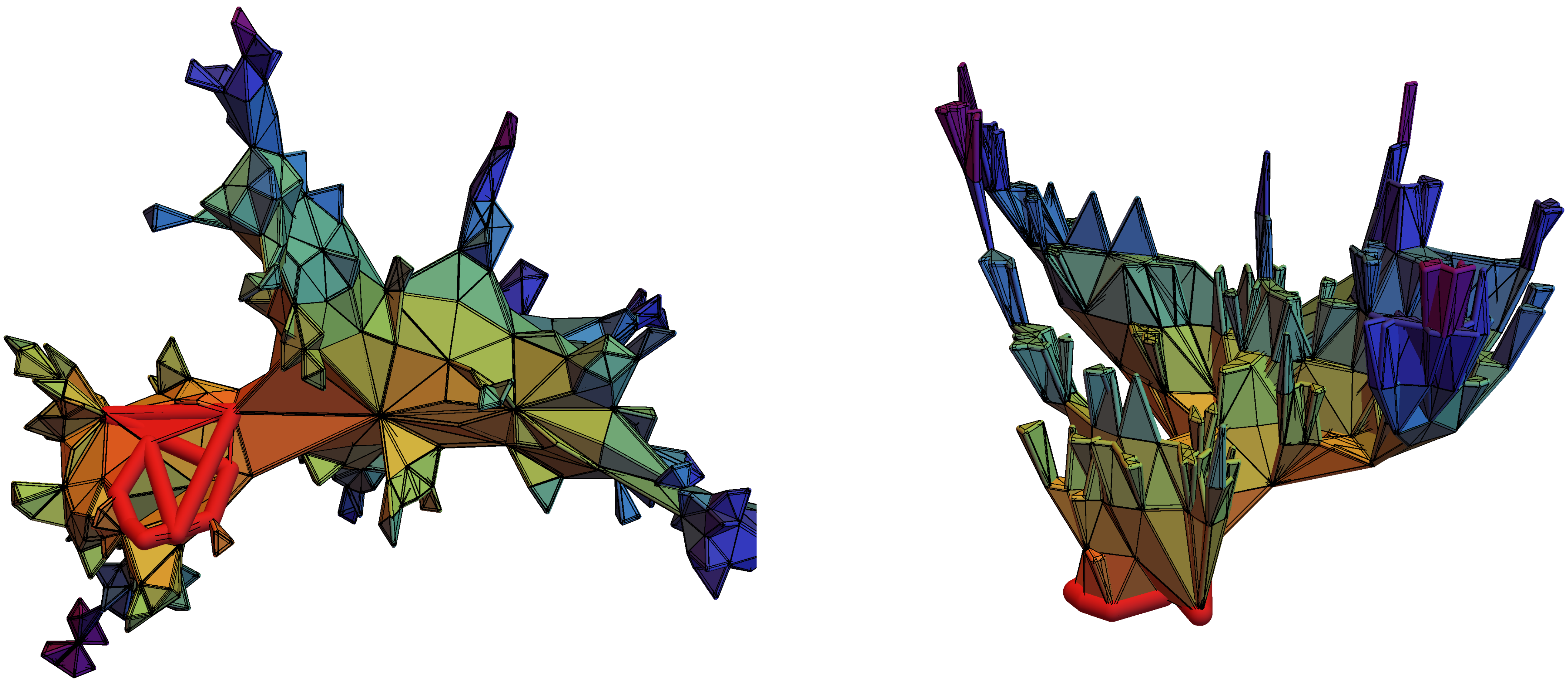

Figure: A Boltzmann triangulation of the 9-gon.

Level sets of Boltzmann triangulations with a boundary

Large Boltzmann triangulations with a boundary

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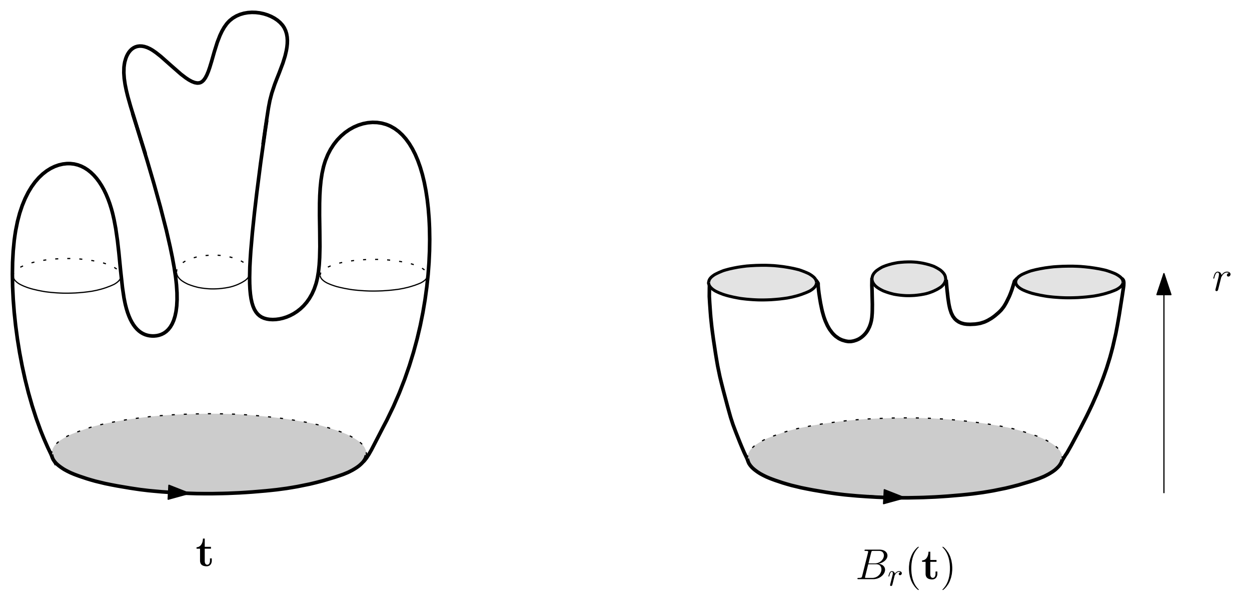

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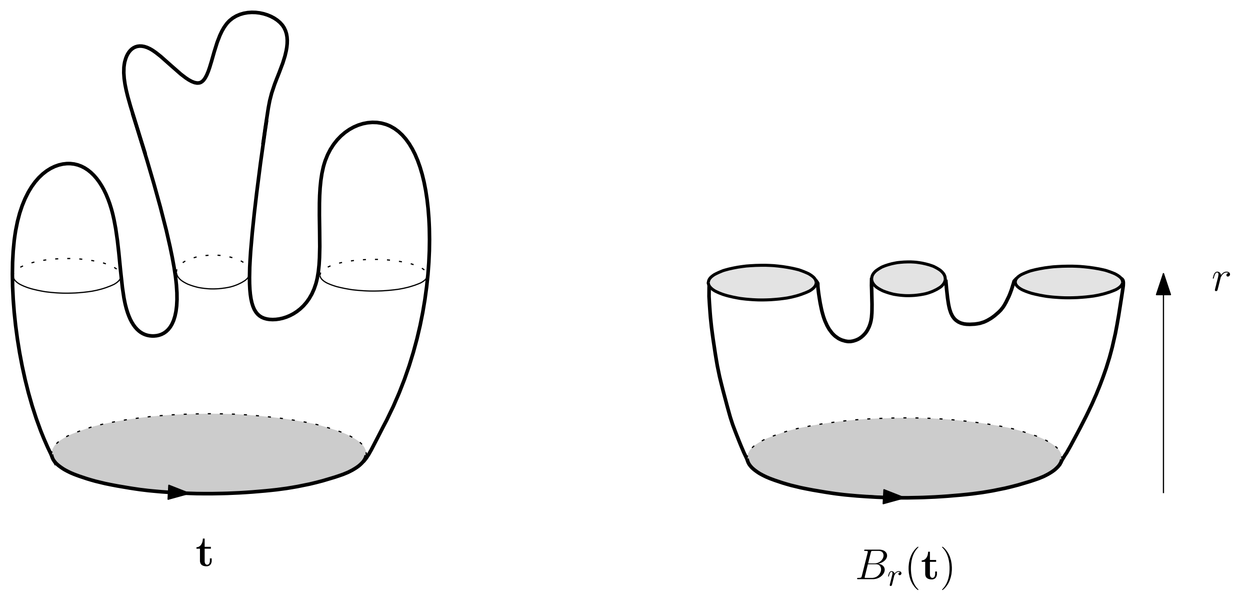
Let $\mathsf{T}^{(p)}$ be a random Boltzmann triangulation of the p-gon, let $B_r(\mathsf{T}^{(p)})$ be the map made of the vertices with distance at most r from the boundary, and

$$\mathbb{L}^{(p)}(r) := \left(L_1^{(p)}(r), L_2^{(p)}(r), \ldots\right).$$

be lengths (or perimeters) of the cycles of $B_r(\mathsf{T}^{(p)})$, ranked in decreasing order.



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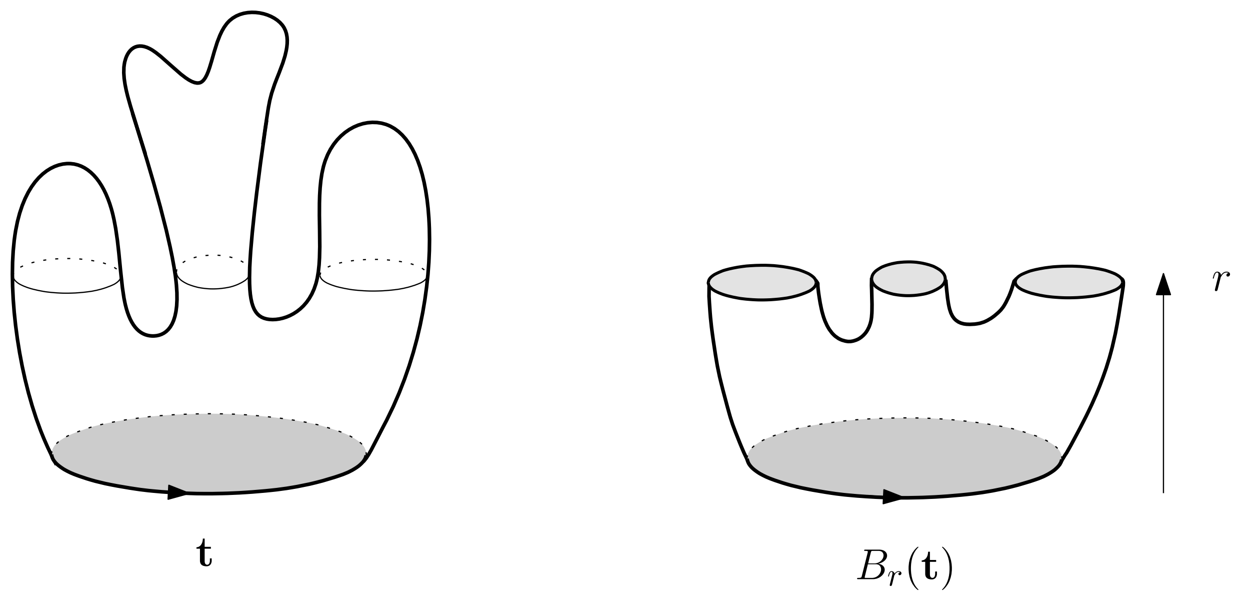

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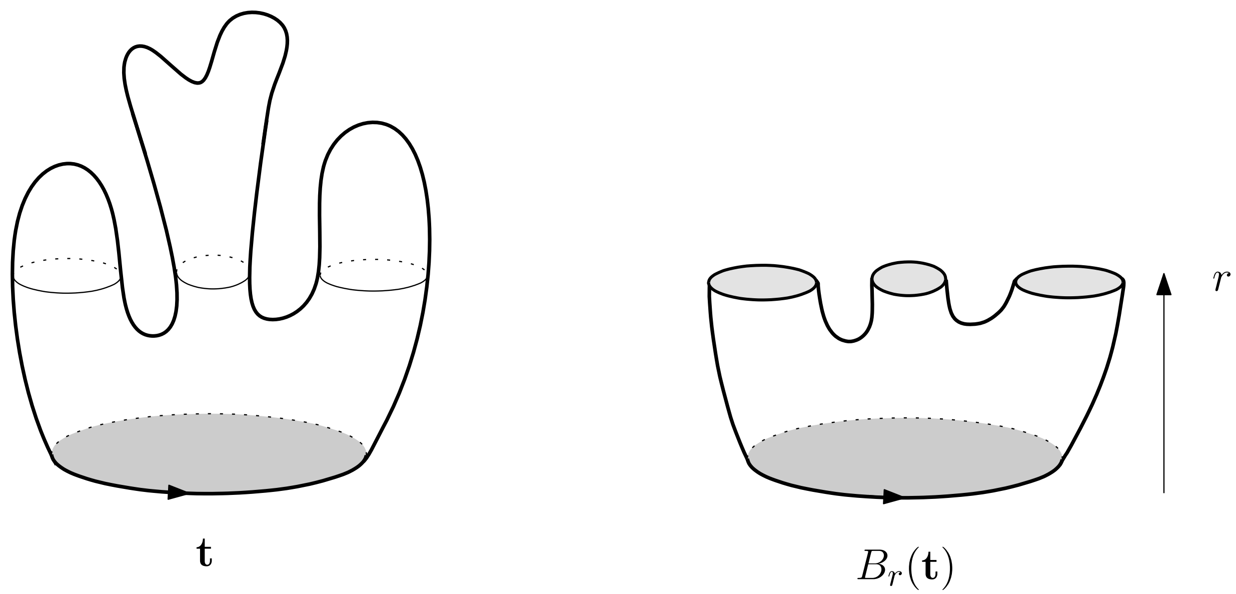

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Simulation

The theorem

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Theorem (Bertoin, Curien, K. '15)**.**

We have

$$\left(\frac{1}{p} \cdot \mathbb{L}^{(p)}(t\sqrt{p}); t \geqslant 0\right) \quad \xrightarrow[p\to\infty]{(d)} \quad \left(\mathbb{X}\left(\frac{3}{2\sqrt{\pi}} \cdot t\right); t \geqslant 0\right),$$

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in distribution in $\ell_3^{\downarrow}$, where $\mathbb{X} = (\mathbb{X}(t); t \geqslant 0)$ is a càdlàg process with values in $\ell_3^{\downarrow}$, which is a *self-similar growth-fragmentation process* (Bertoin '15).

The main tool: a peeling exploration

Geometry of random maps

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Geometry of random maps

Several techniques to study random maps:

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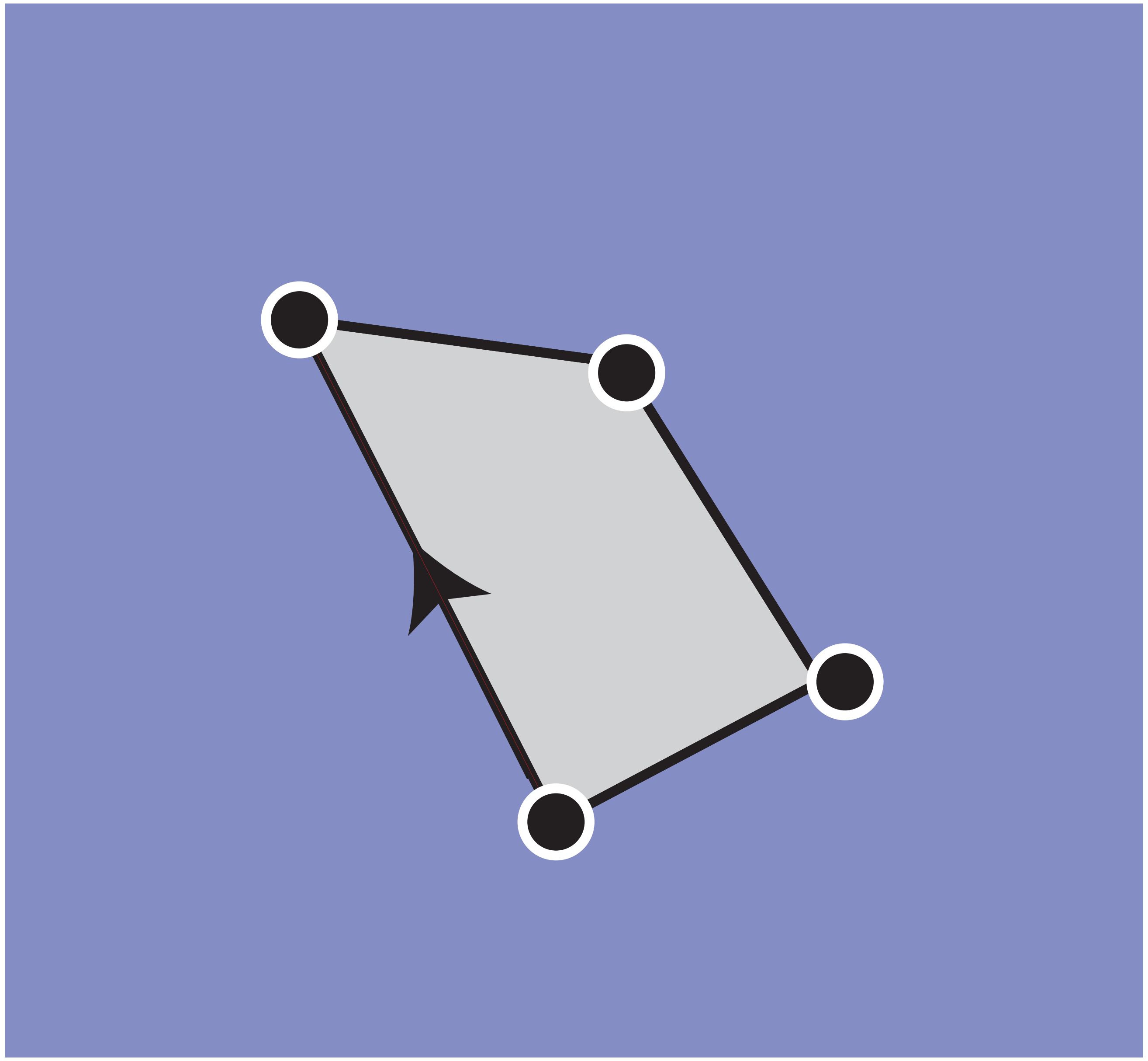
– **peeling**, which is a Markovian way to iteratively explore a random map (Watabiki '95, Angel '03, Budd '14).

Branching peeling

Intuitively, a **branching peeling** of a triangulation with a boundary $\mathfrak{t}$ is an iterative exploration of $\mathfrak{t}$ starting from the boundary and by discovering a new triangle at each step by *peeling an edge* using a deterministic algorithm $\mathcal{A}$.



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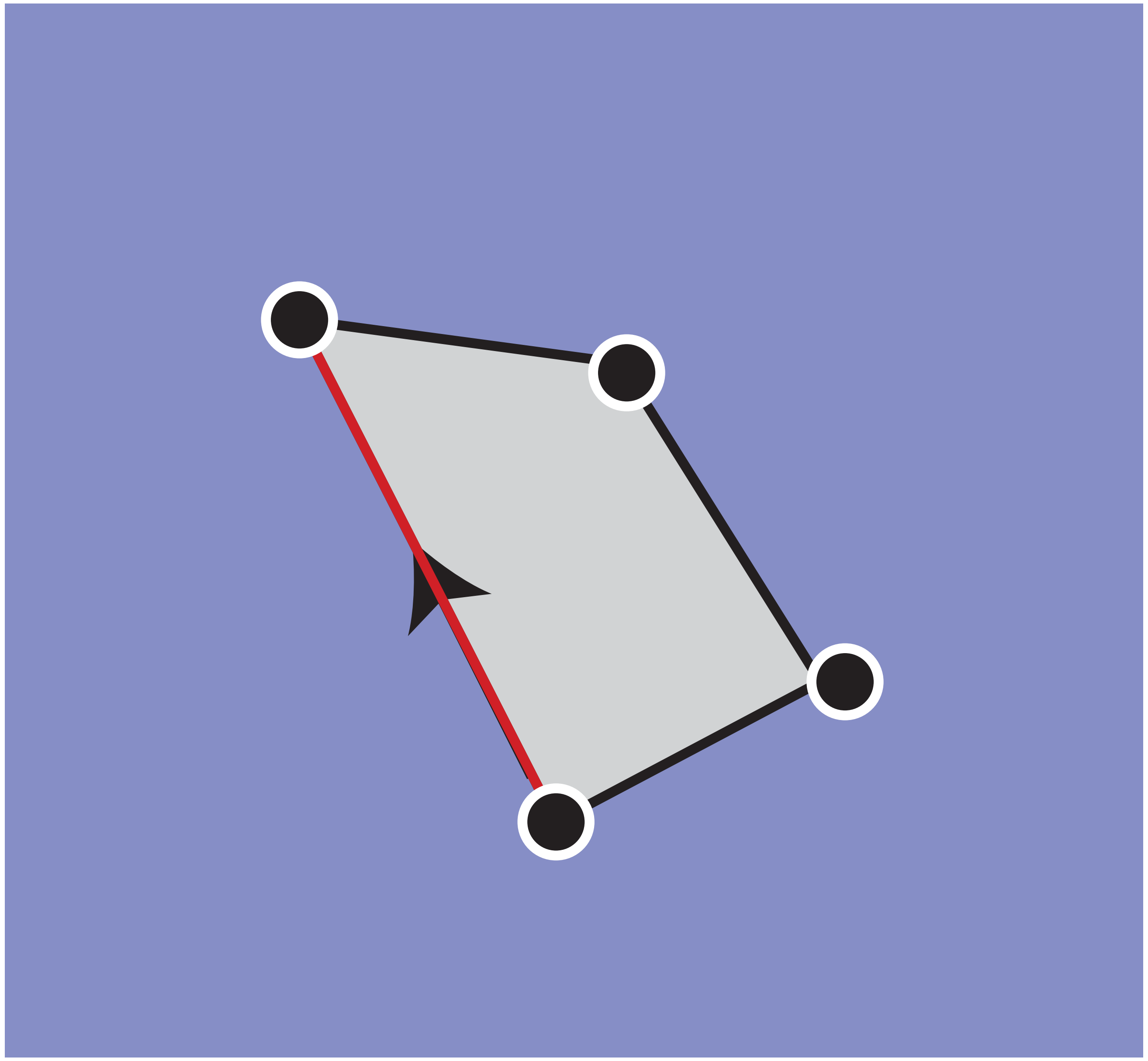

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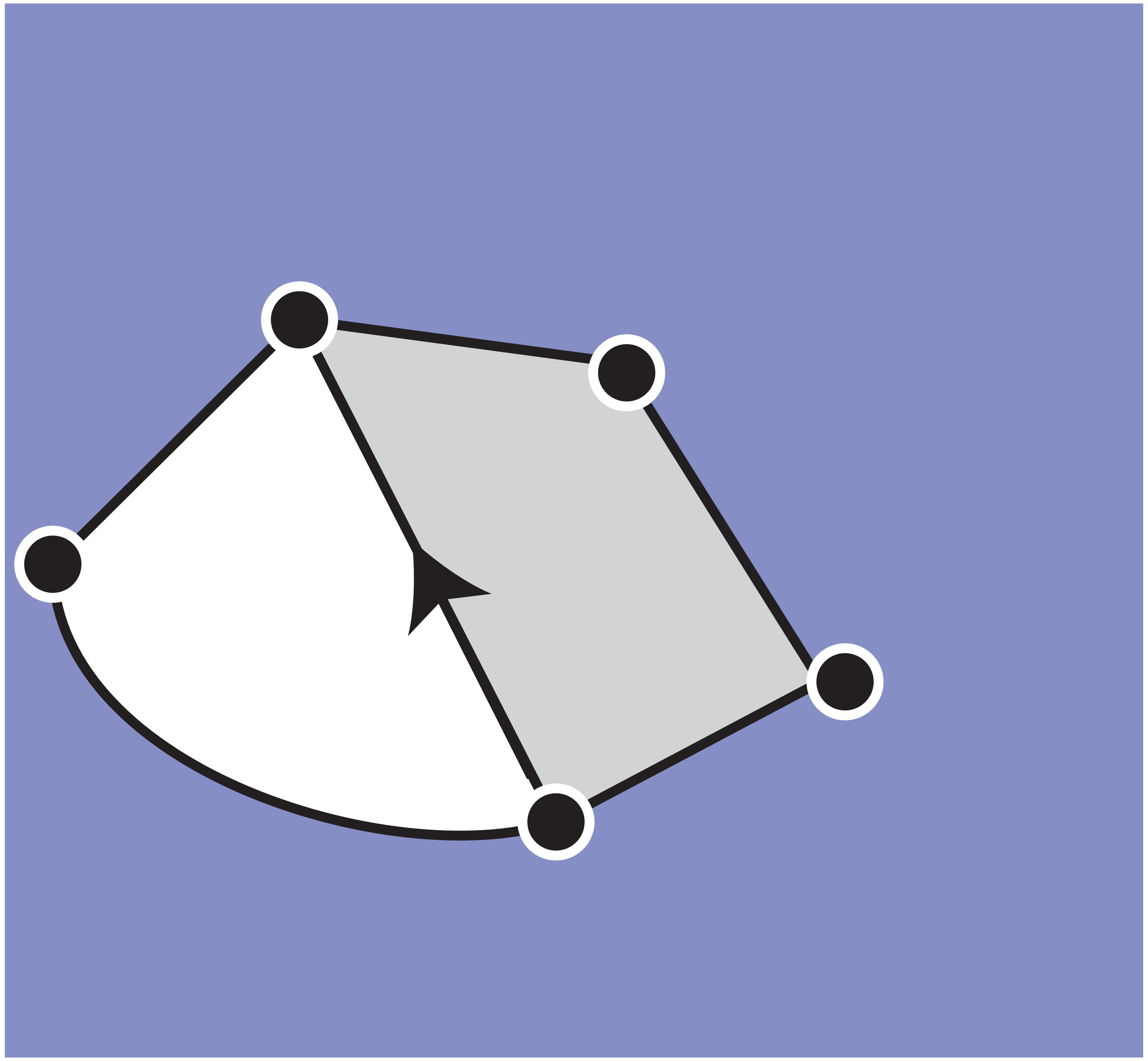

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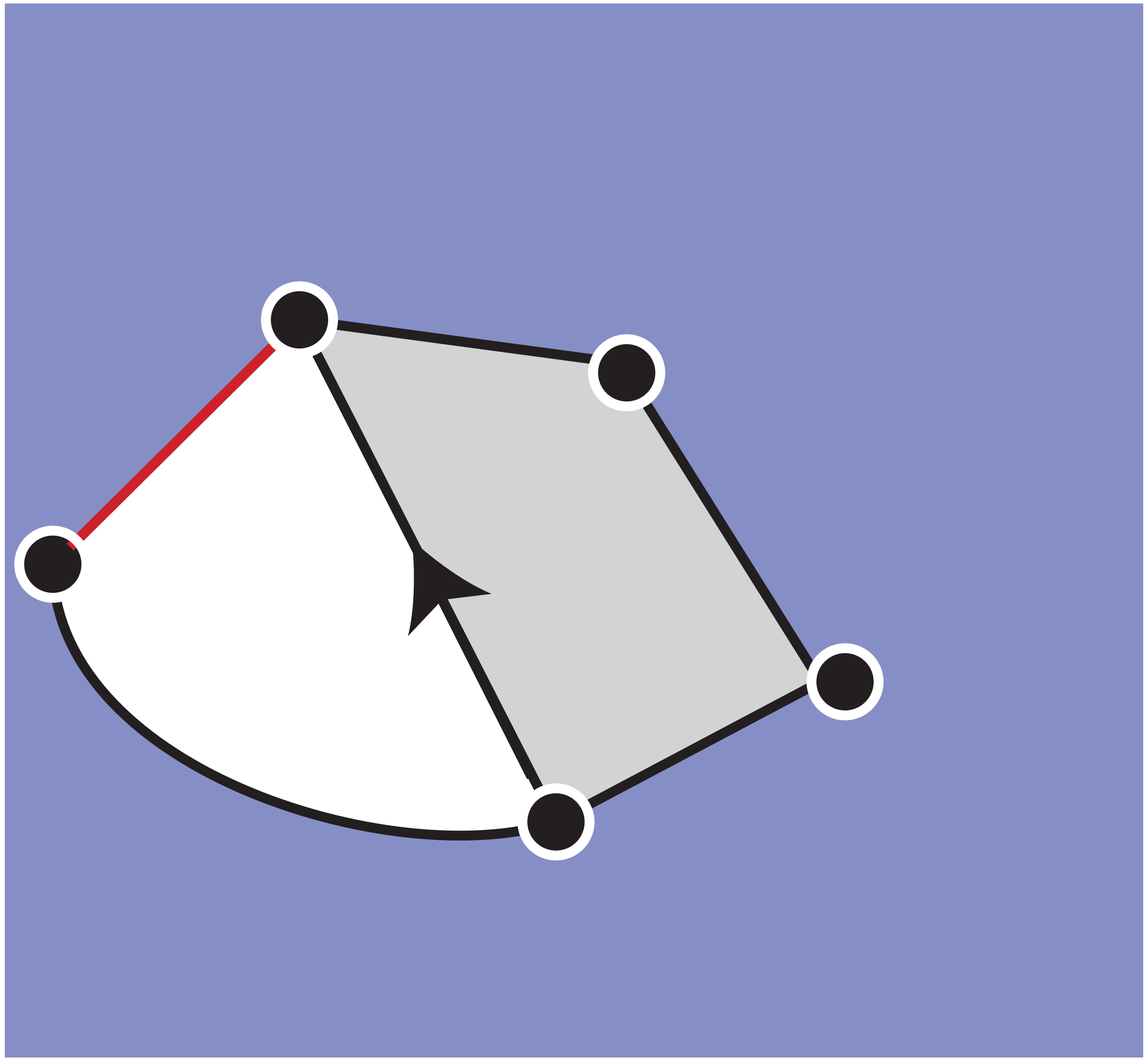

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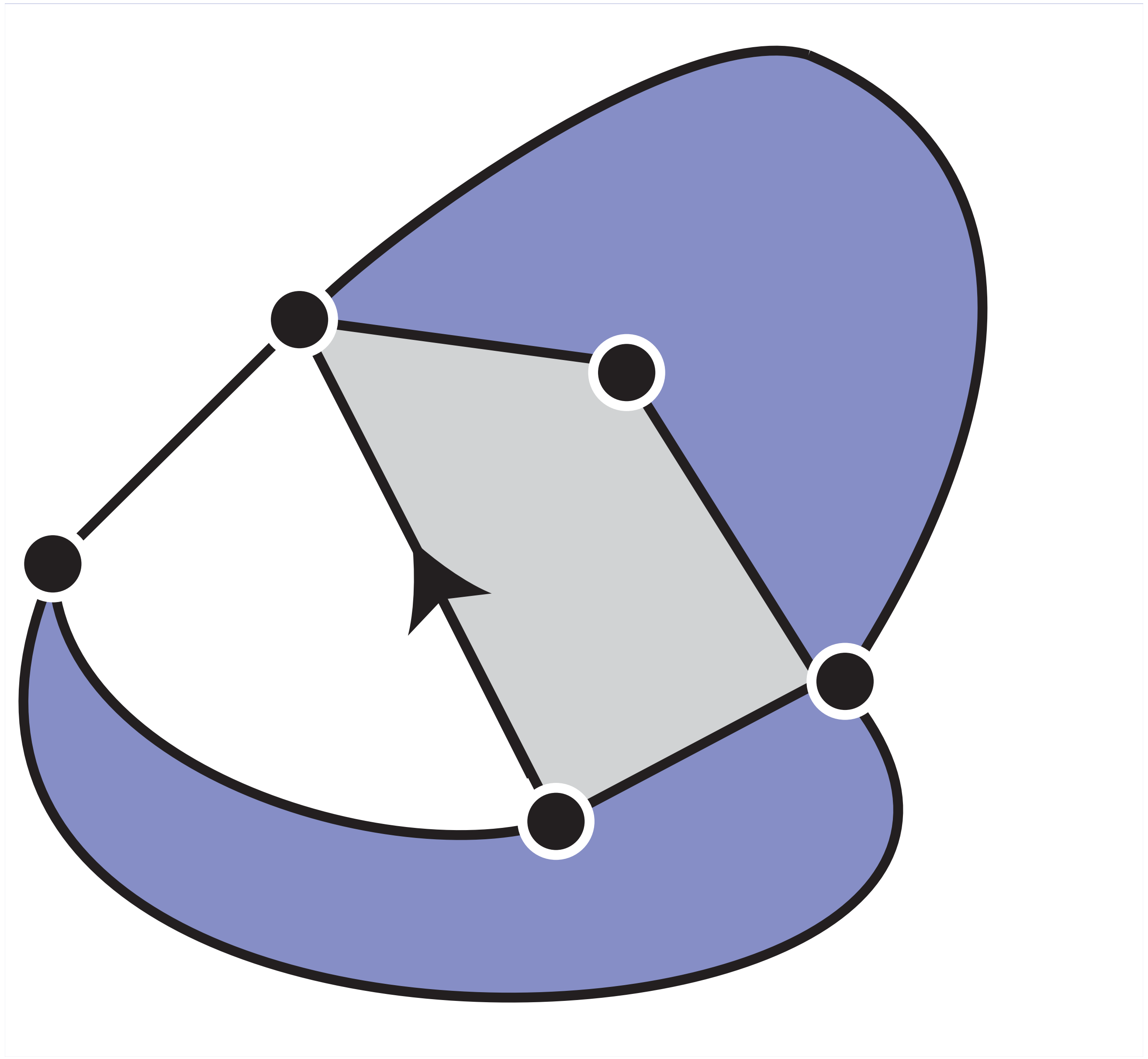

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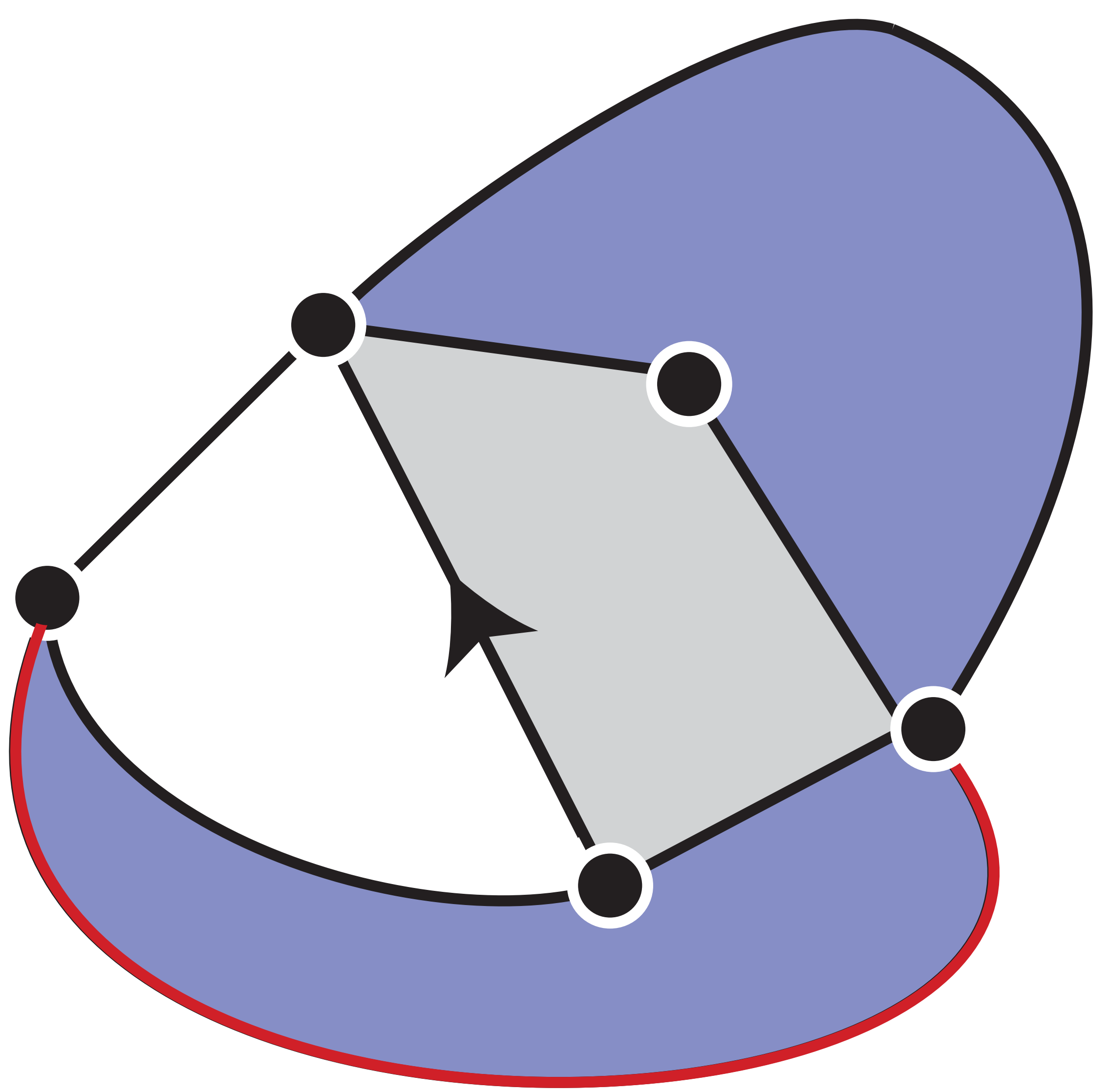

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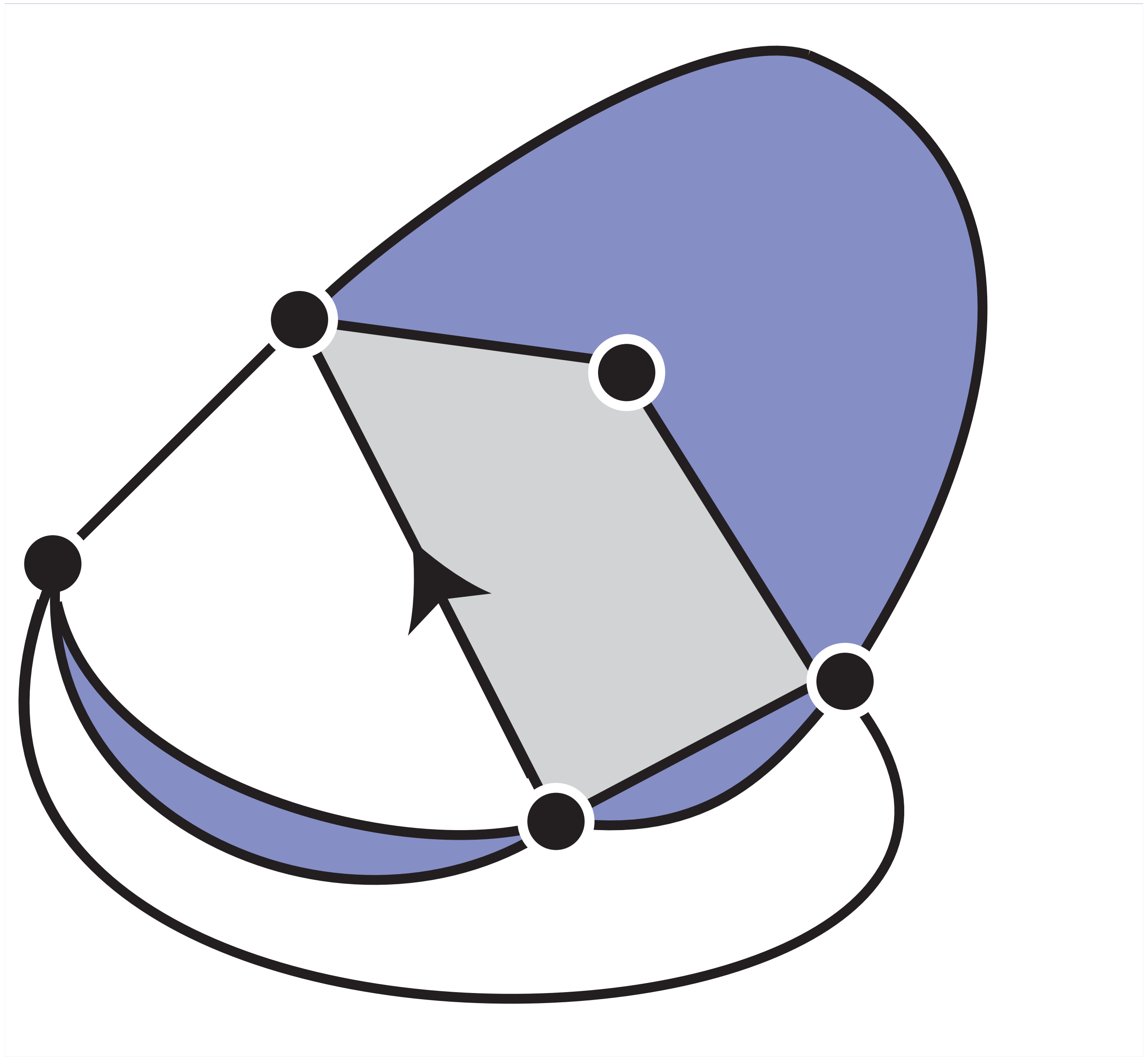

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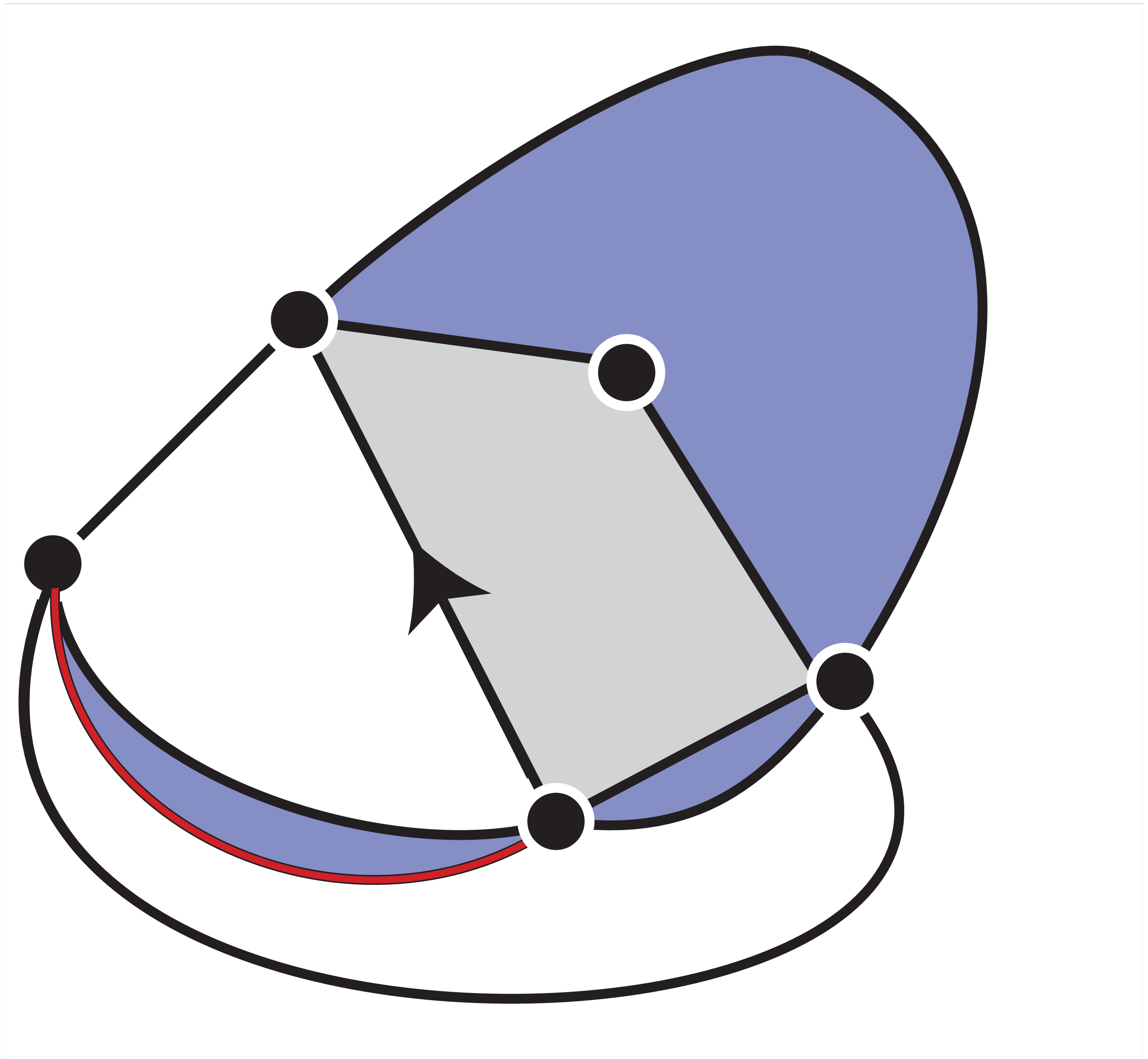

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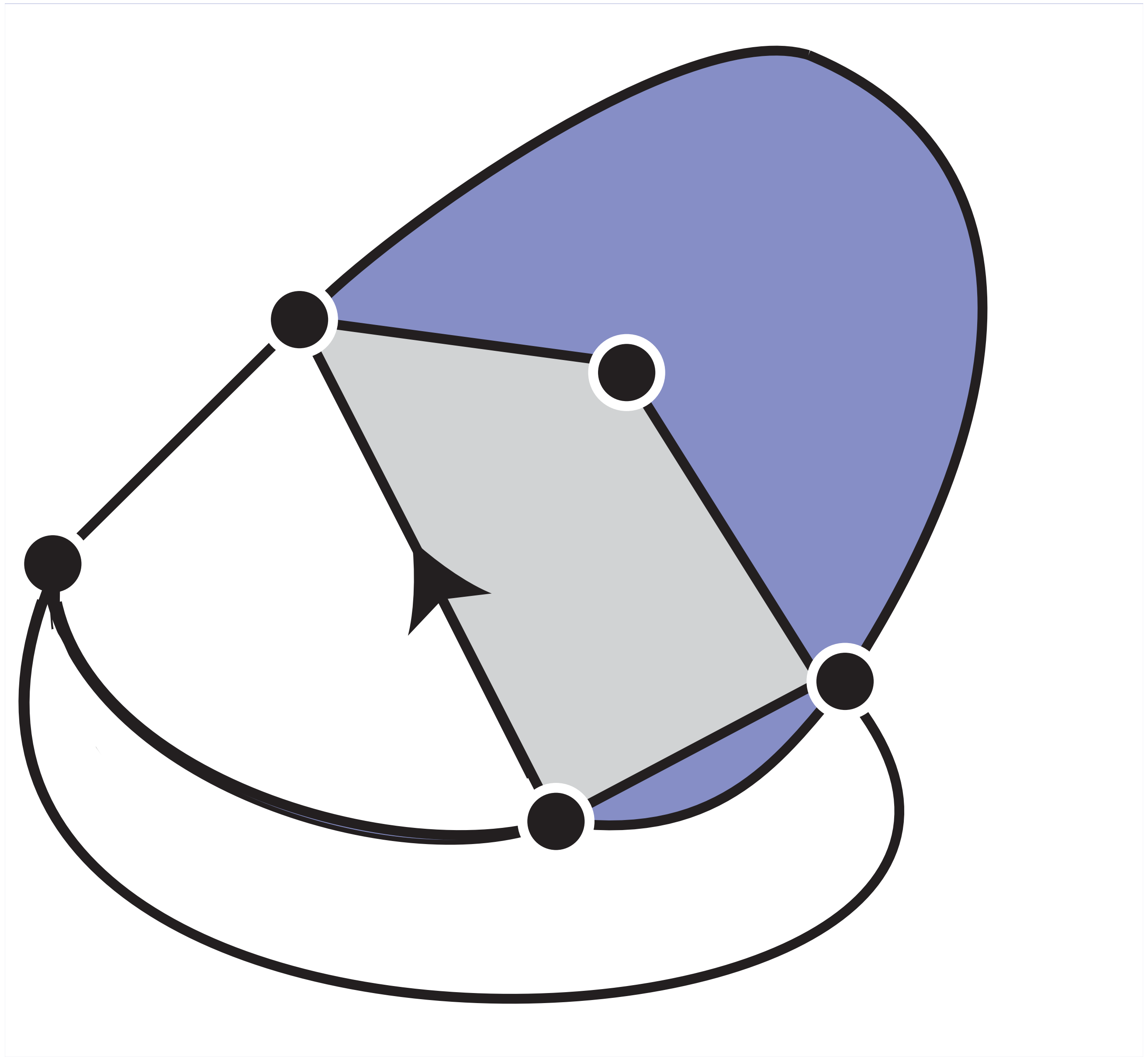

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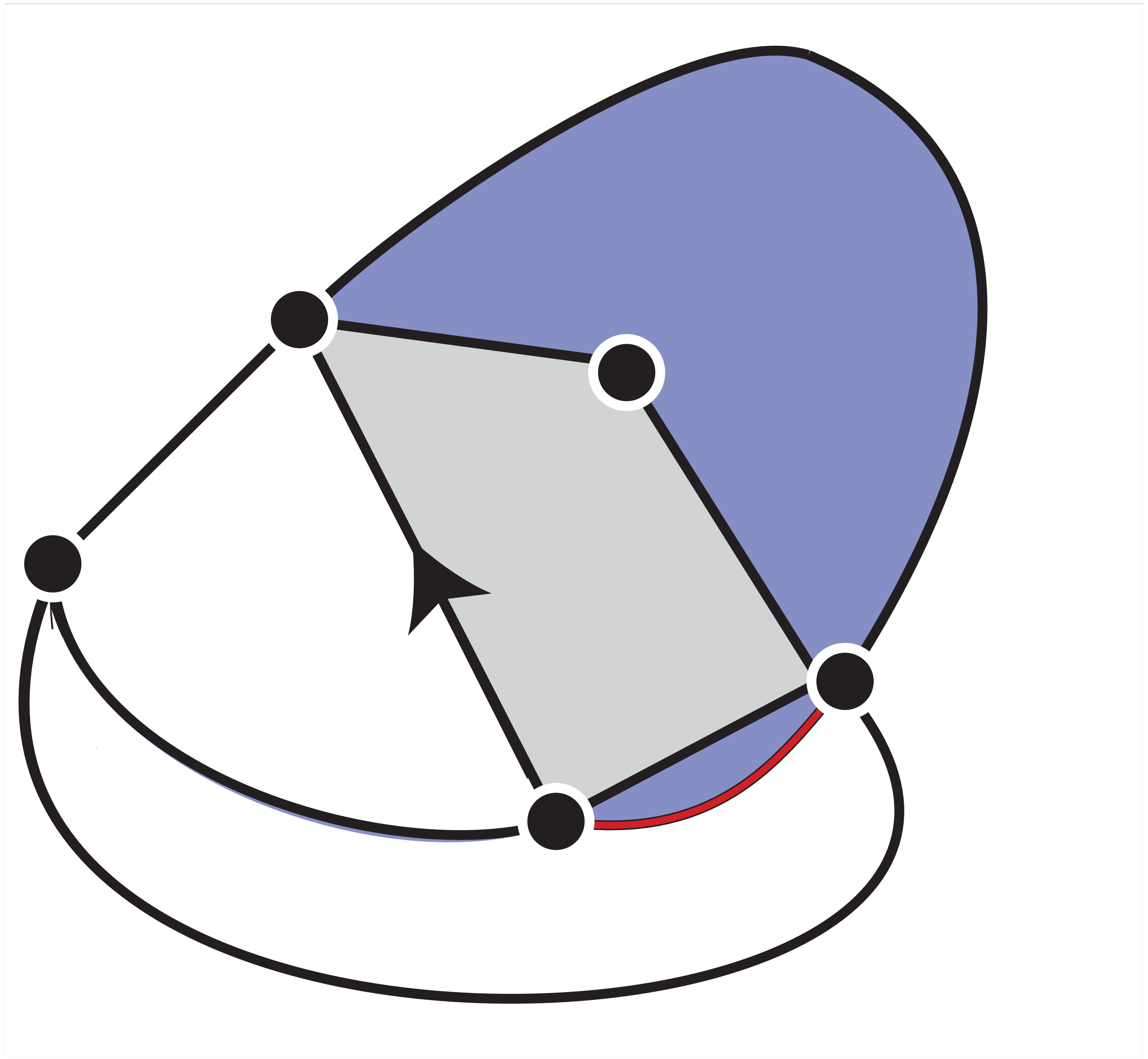

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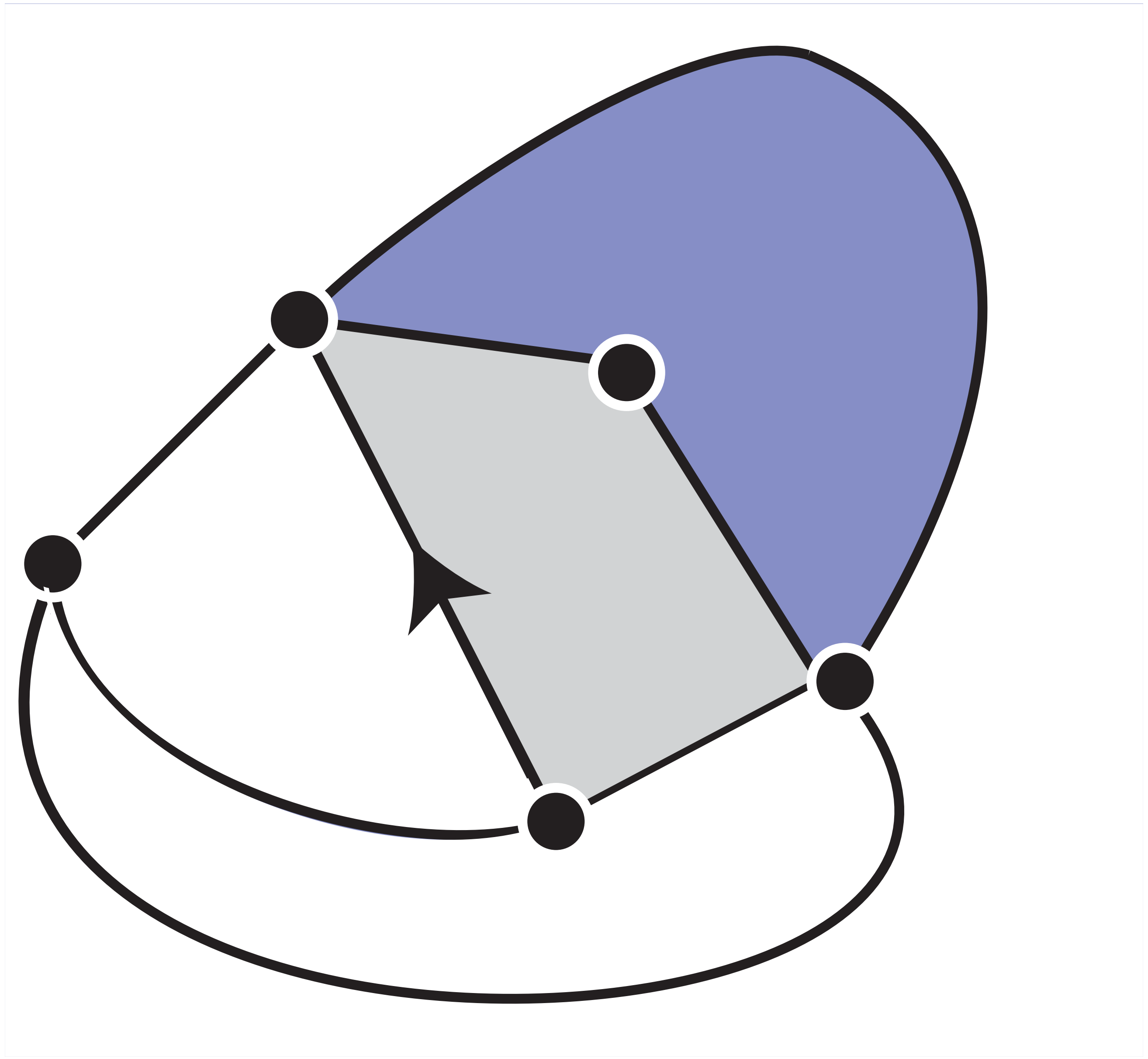

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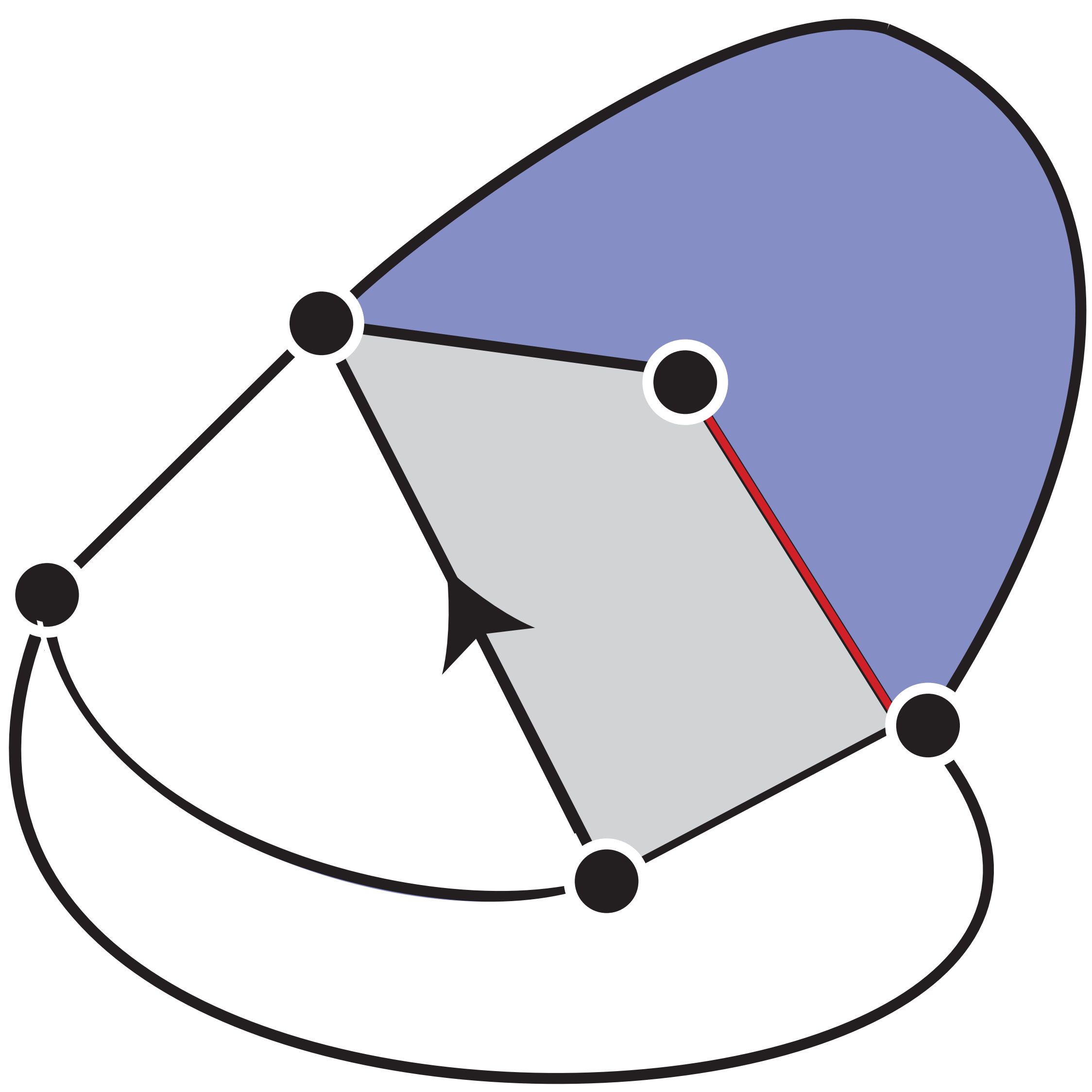

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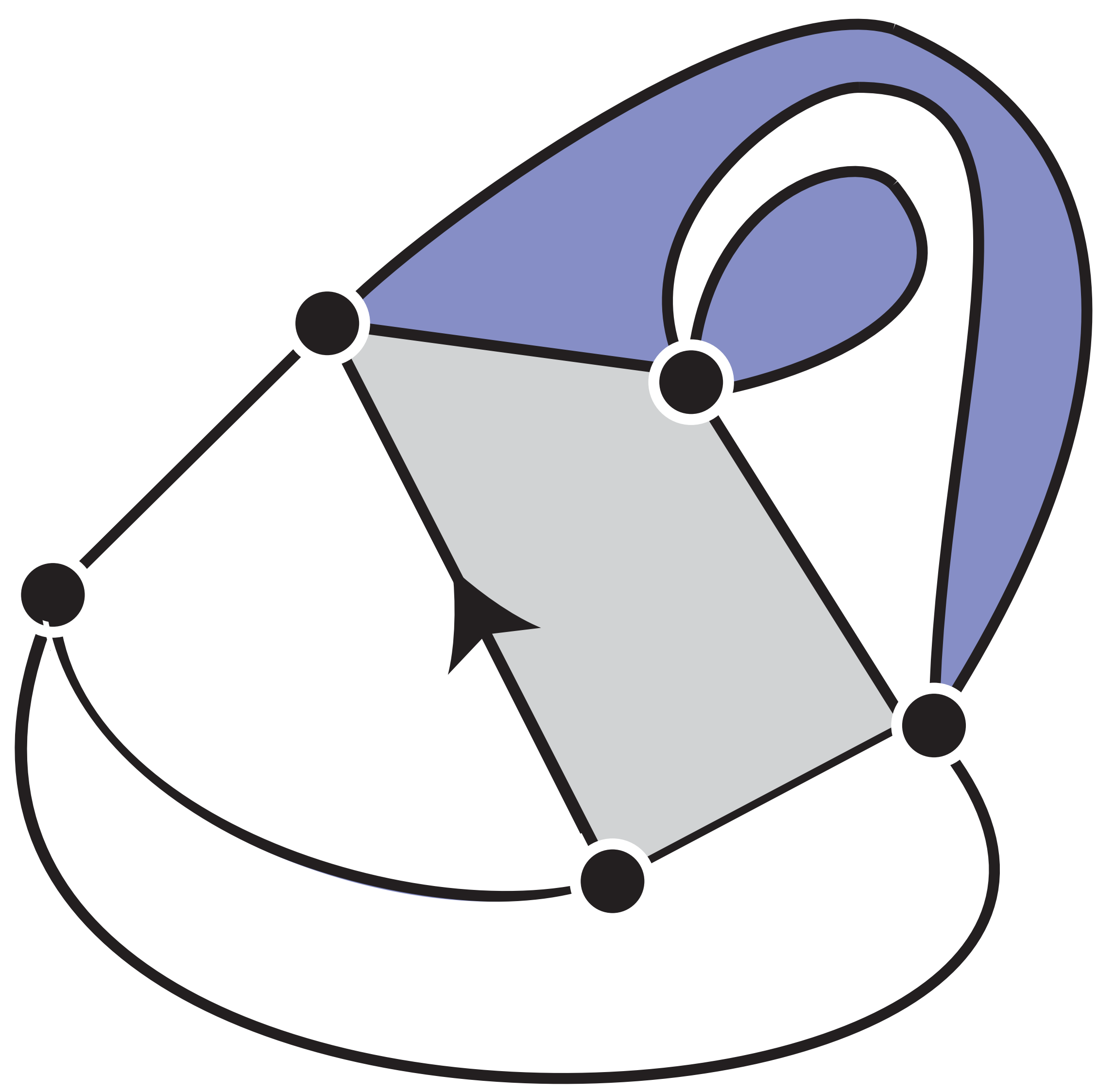

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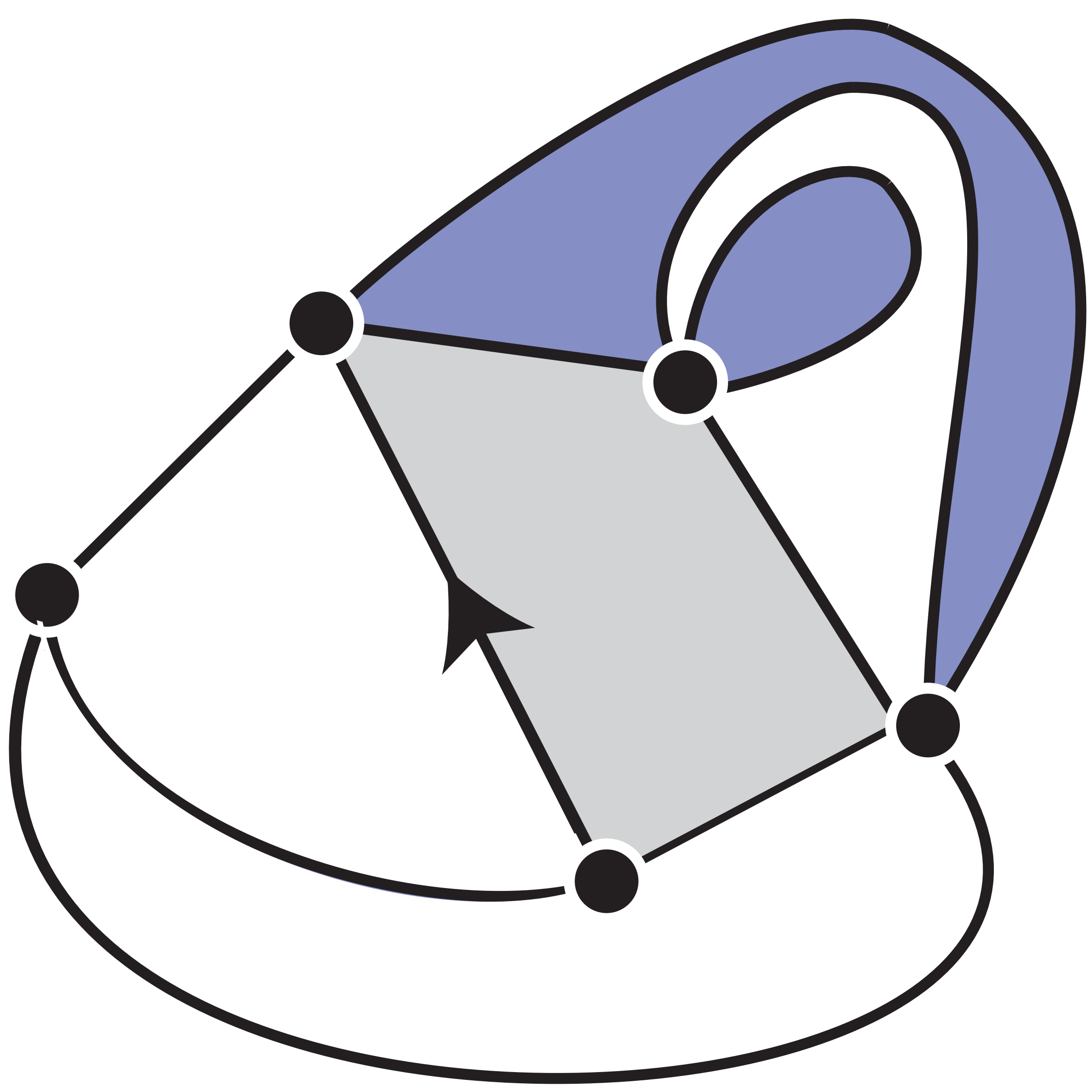
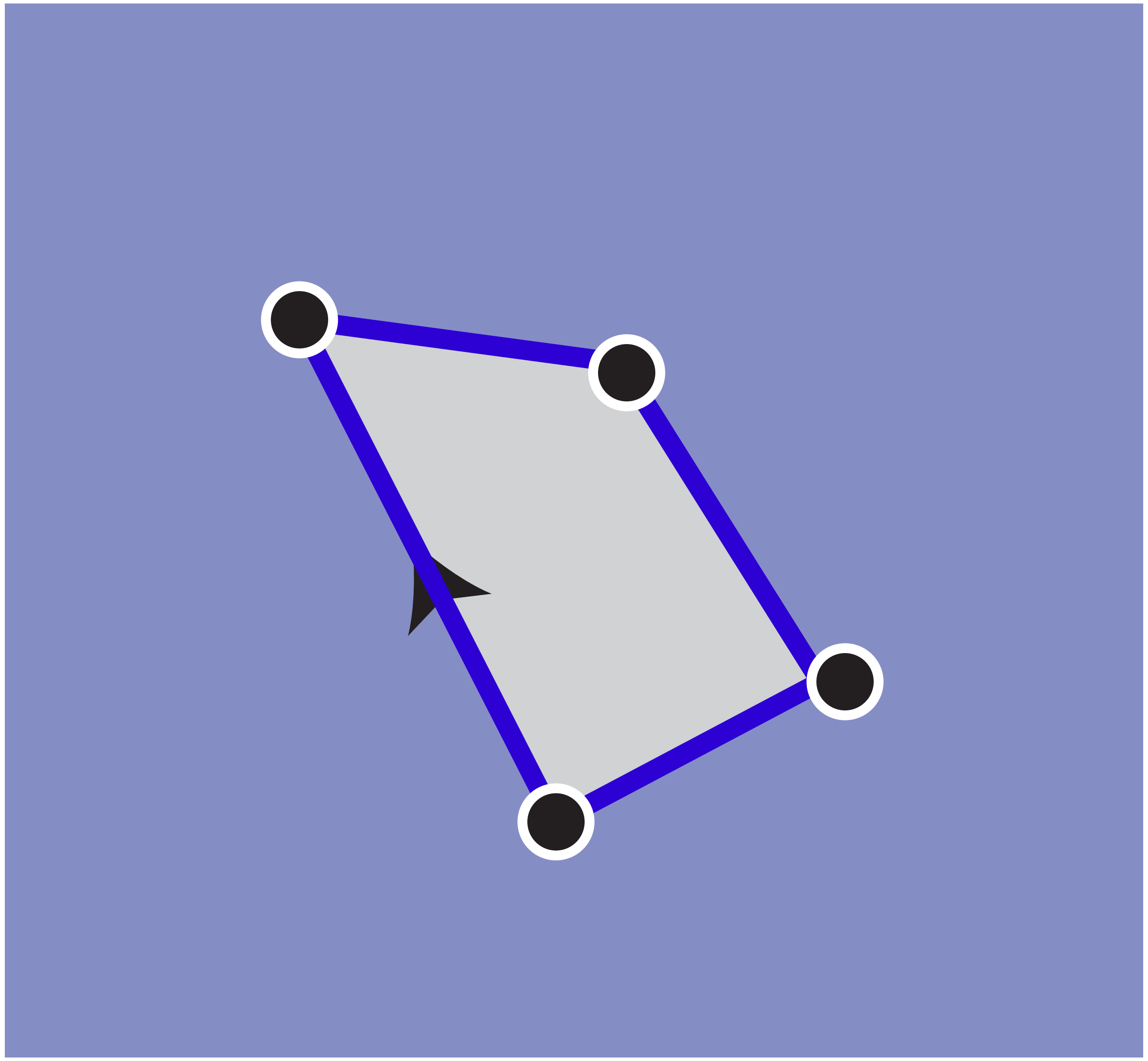


And so on...

Following the locally largest cycle

↝ Idea: at each peeling step, peel along the current locally largest cycle. Let $\widetilde{L}^{(p)}(i)$ its length after i peeling steps.



$\widetilde{L}^{(4)}(0) = 4$

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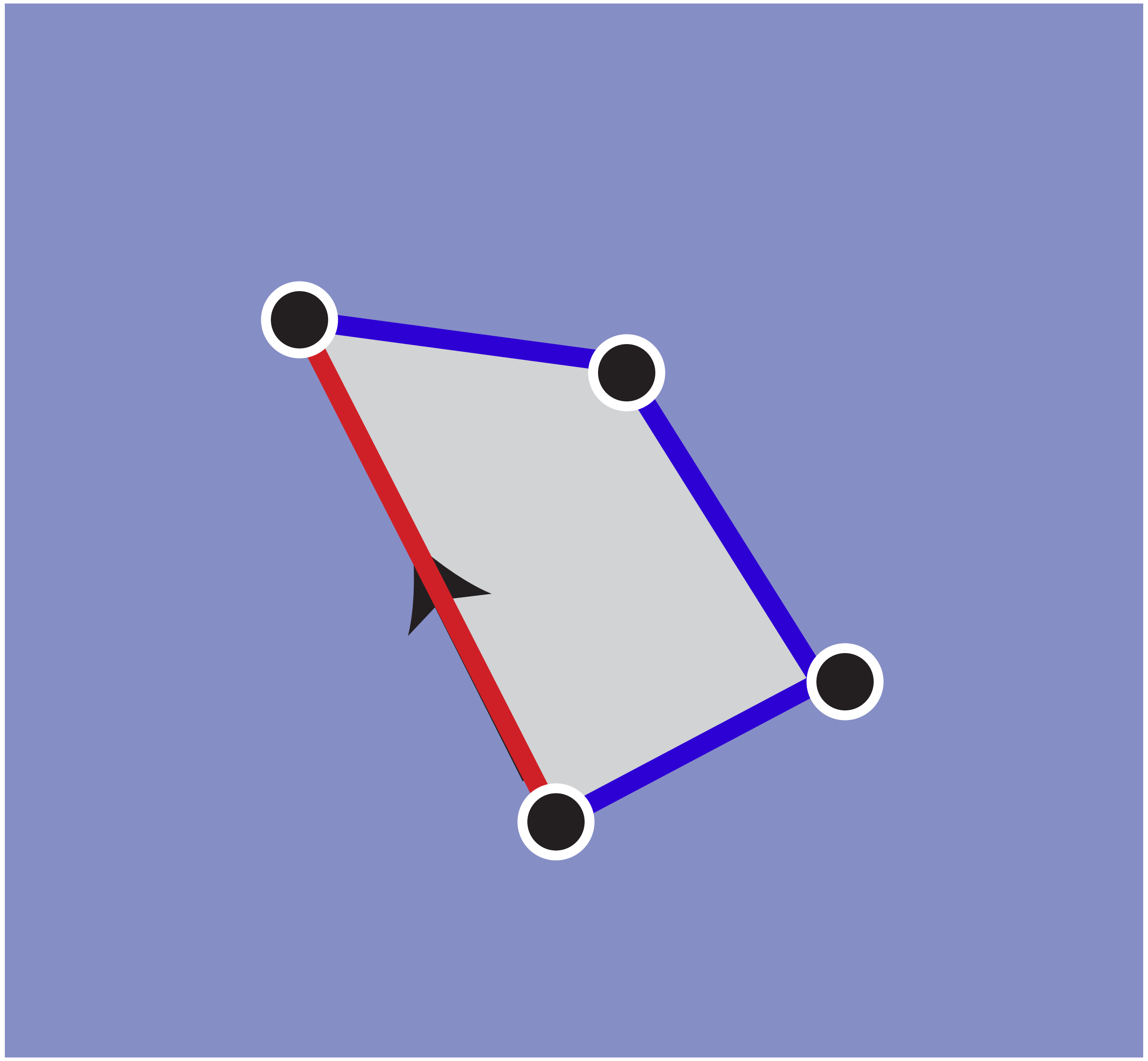

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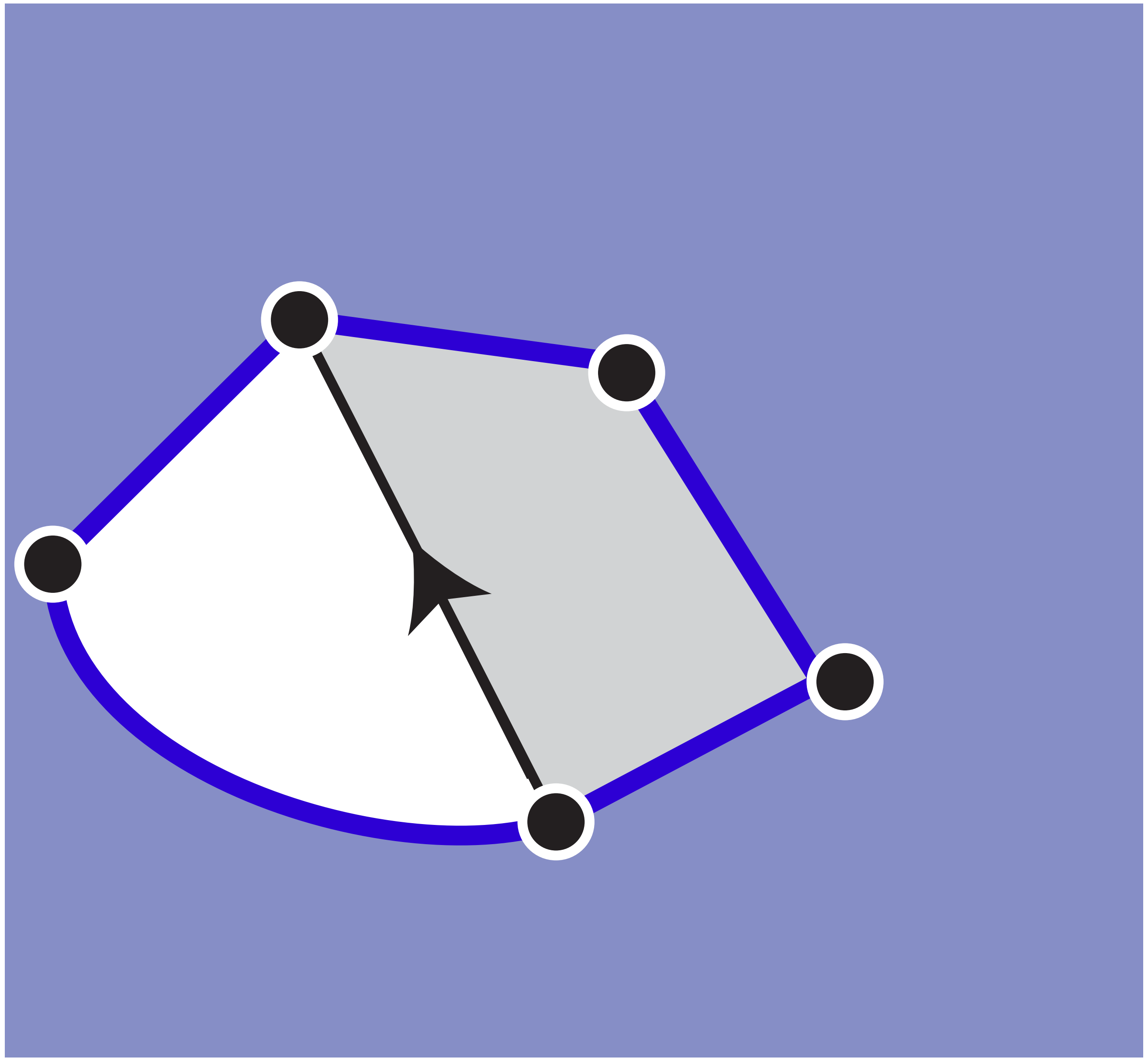

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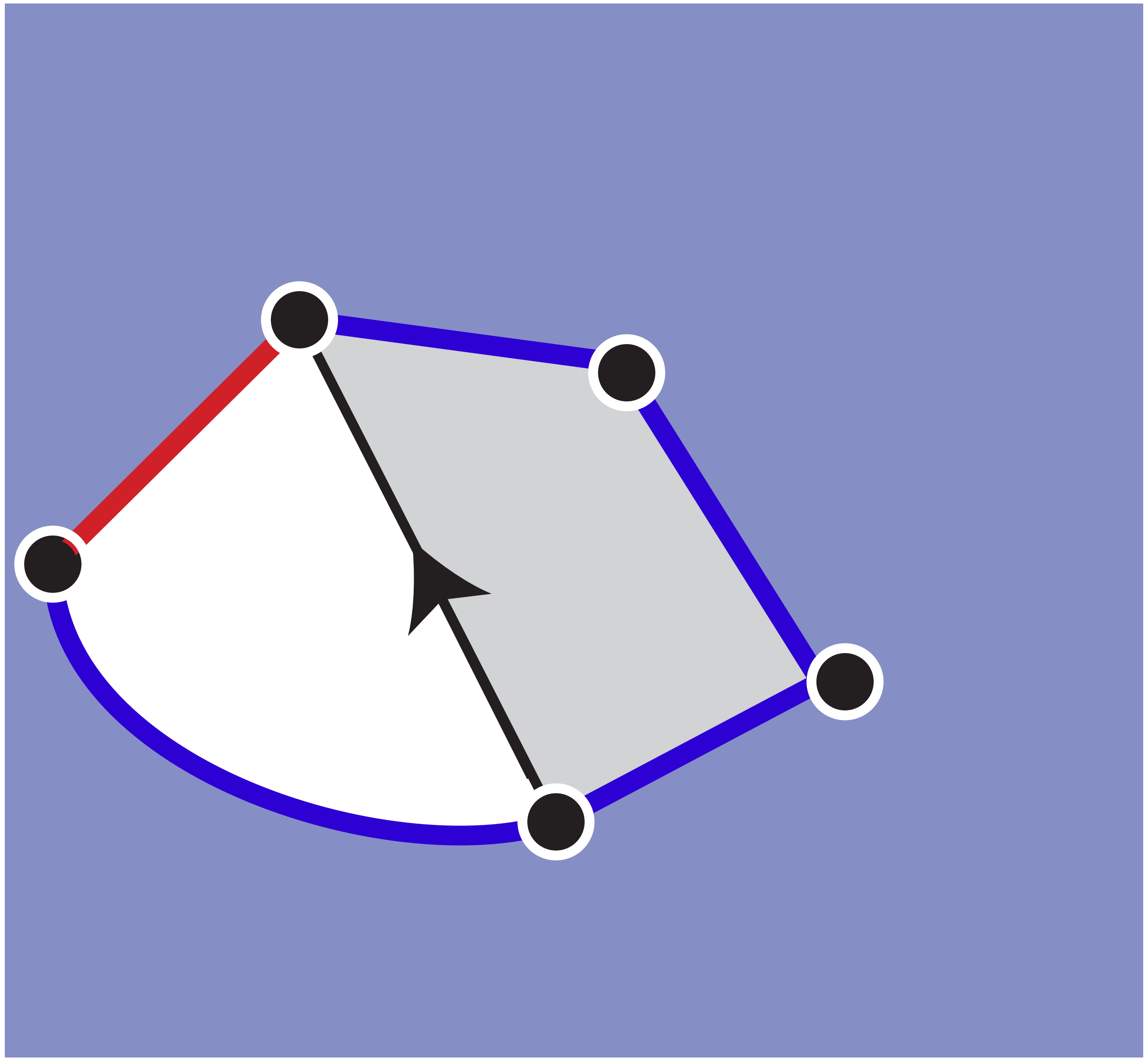

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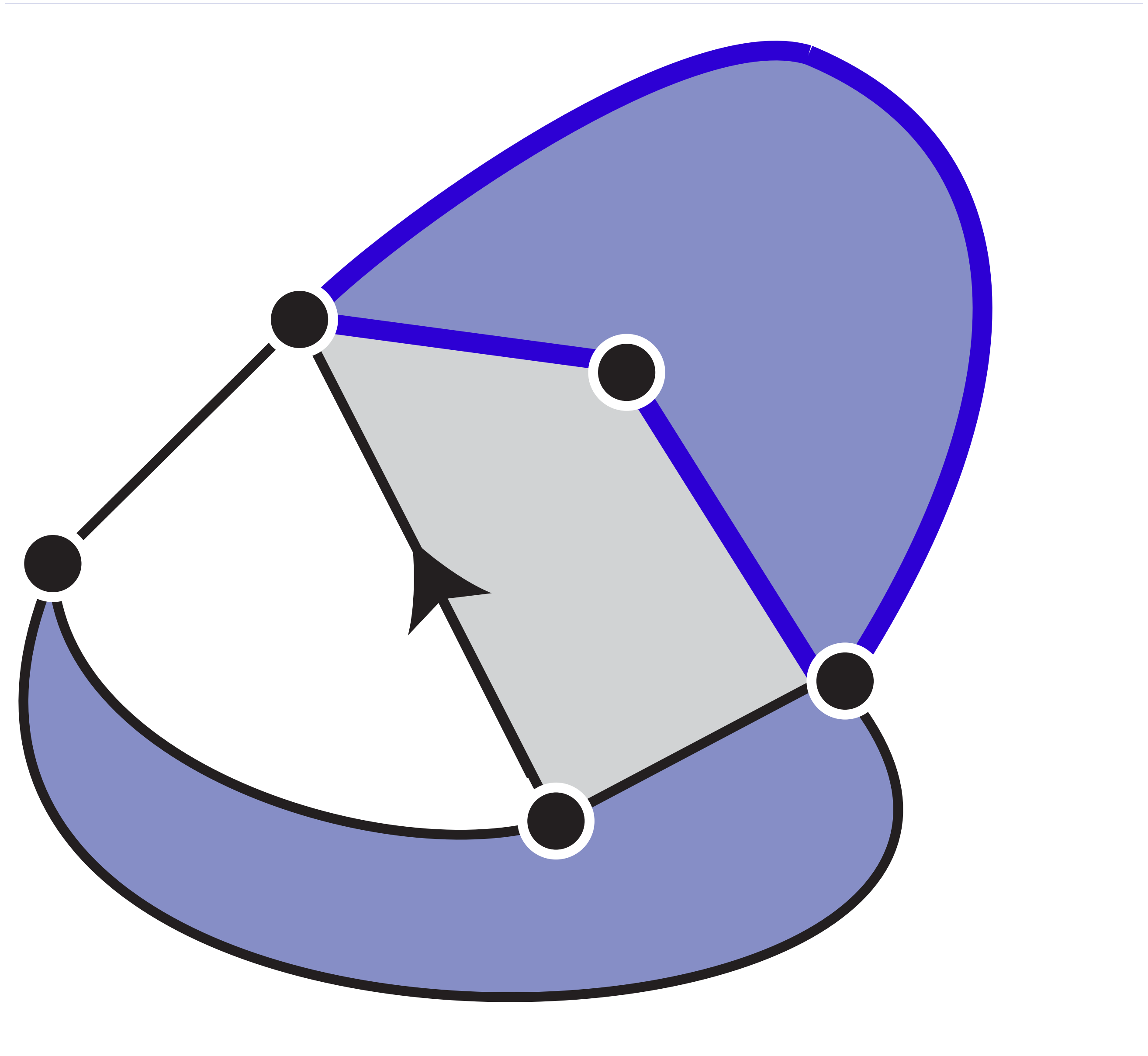

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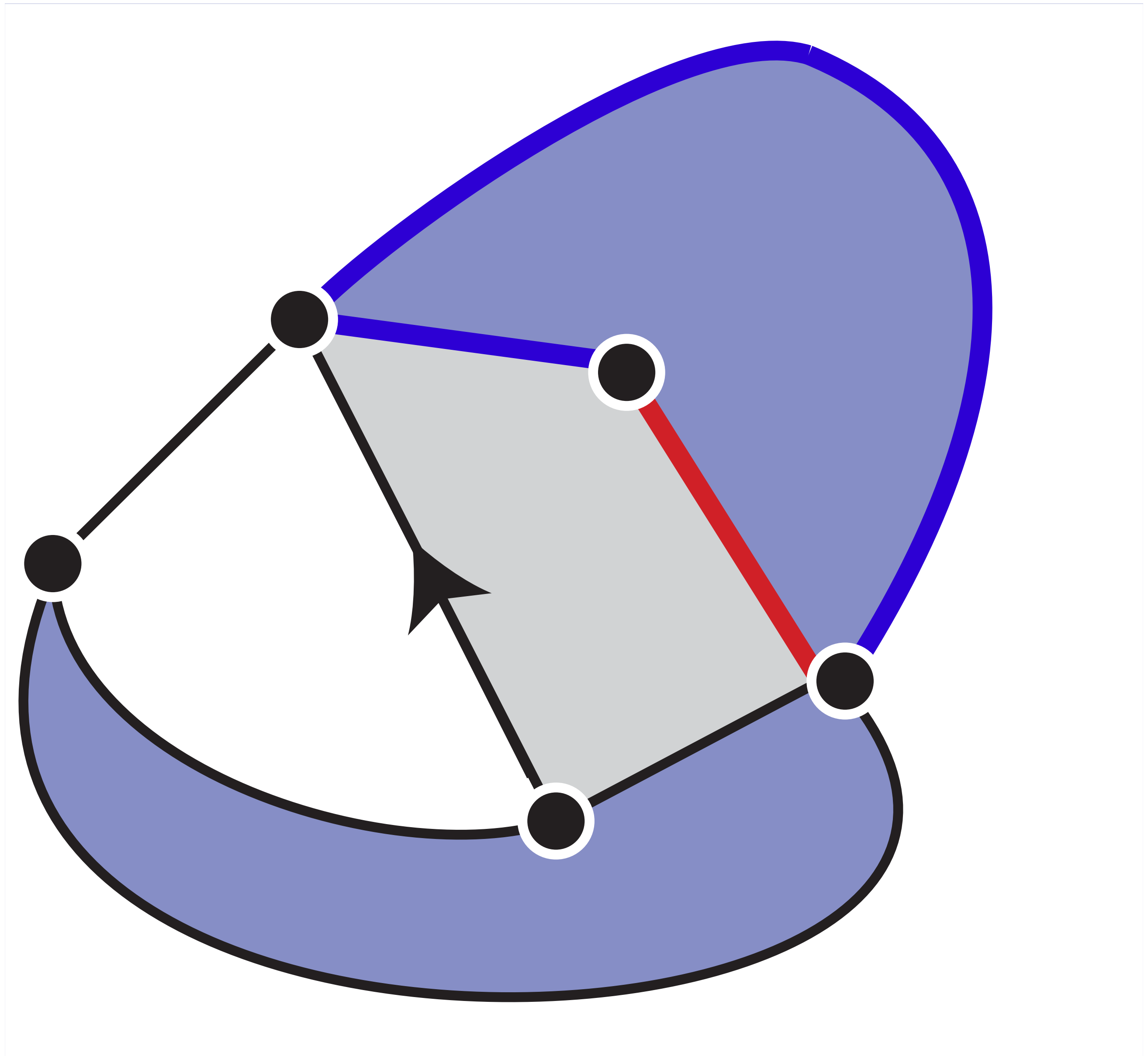

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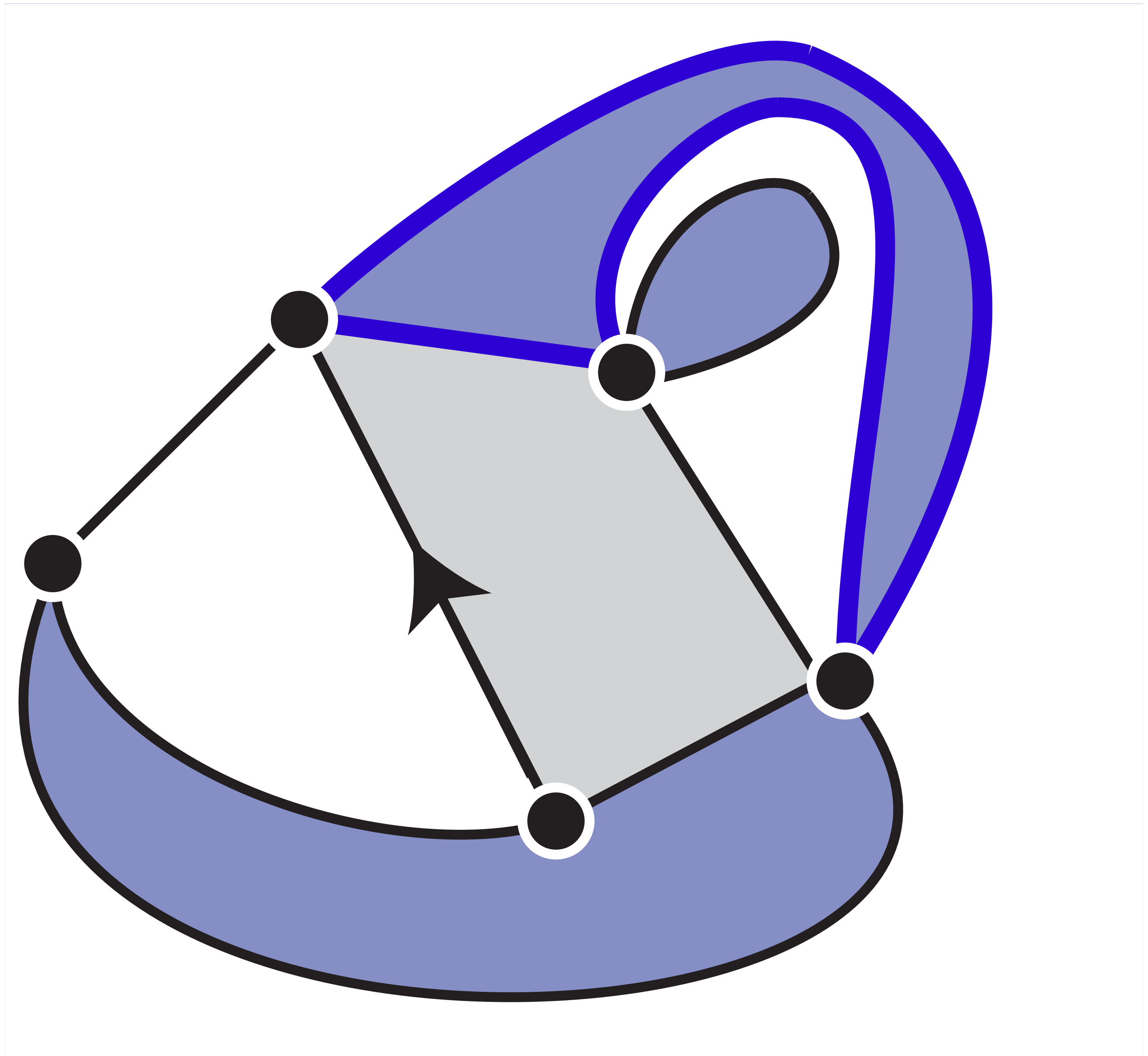

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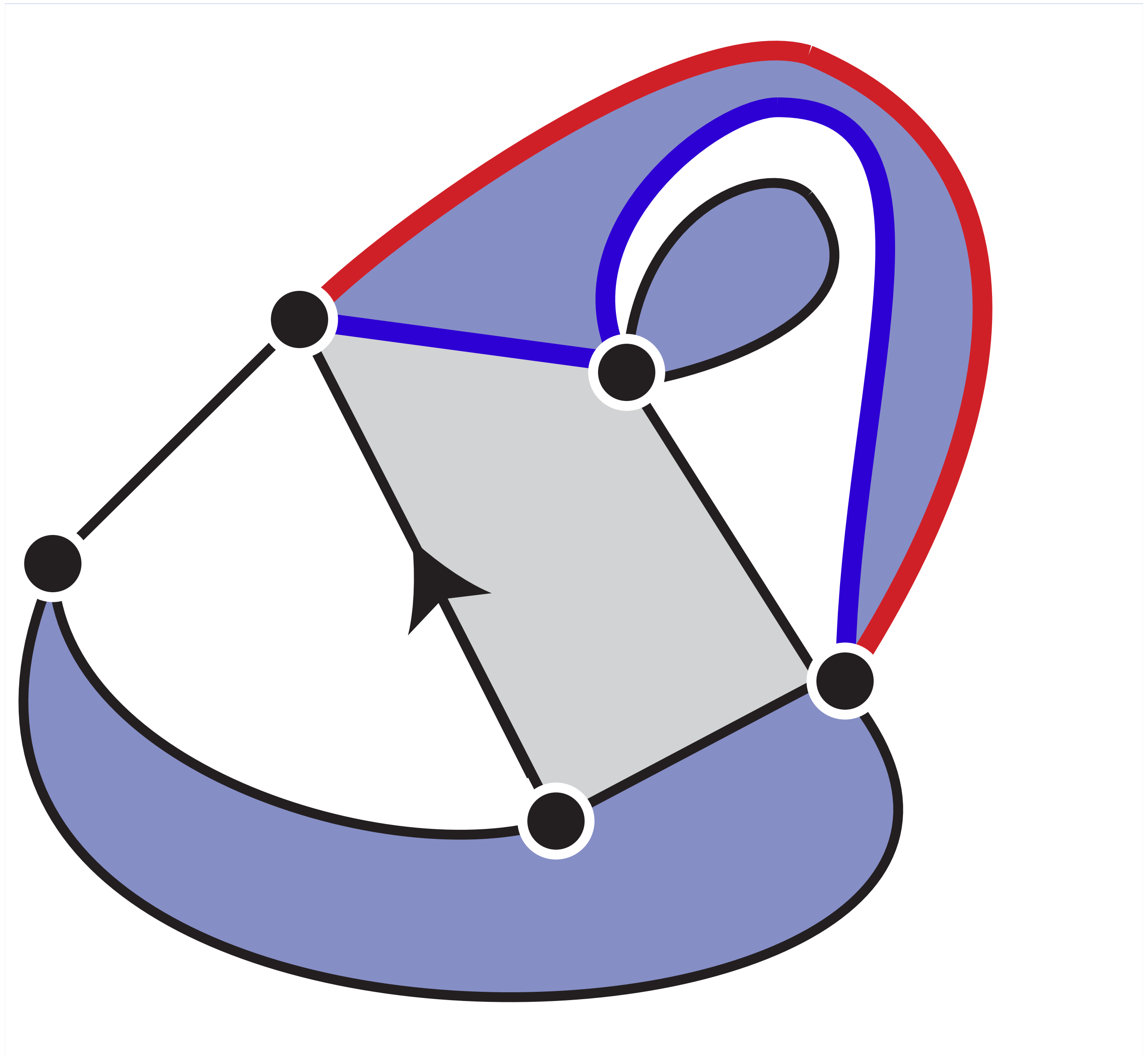

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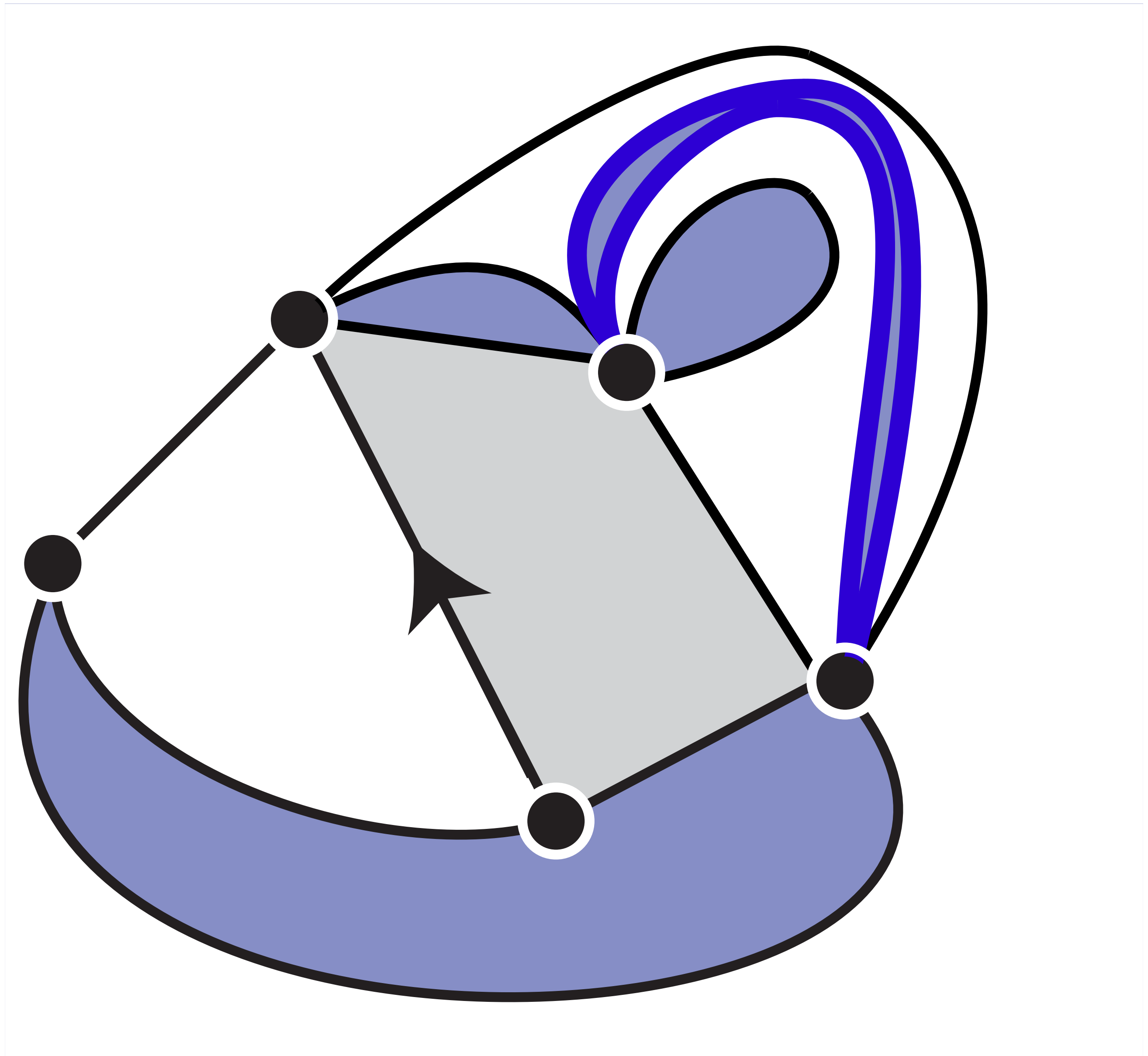

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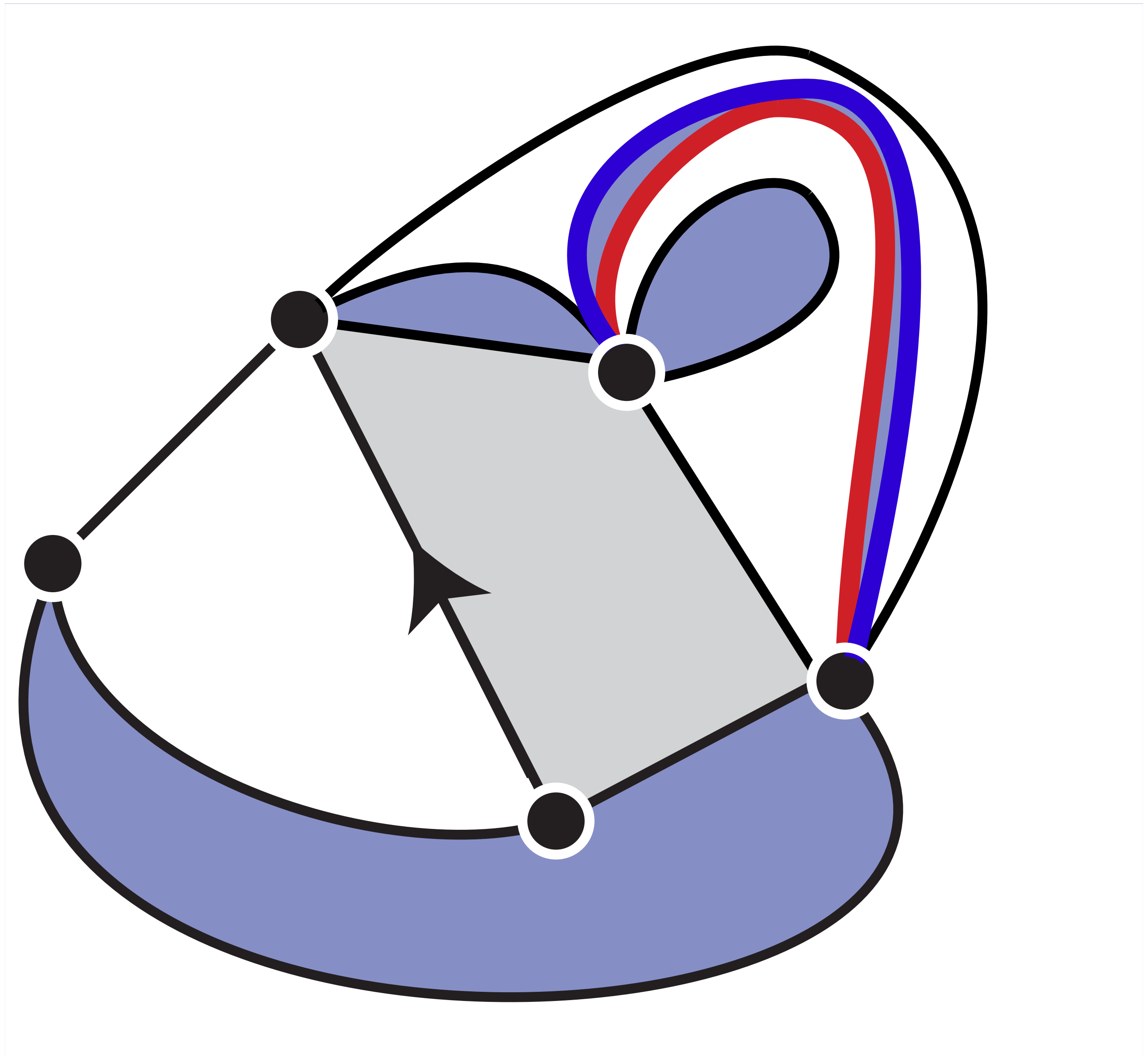

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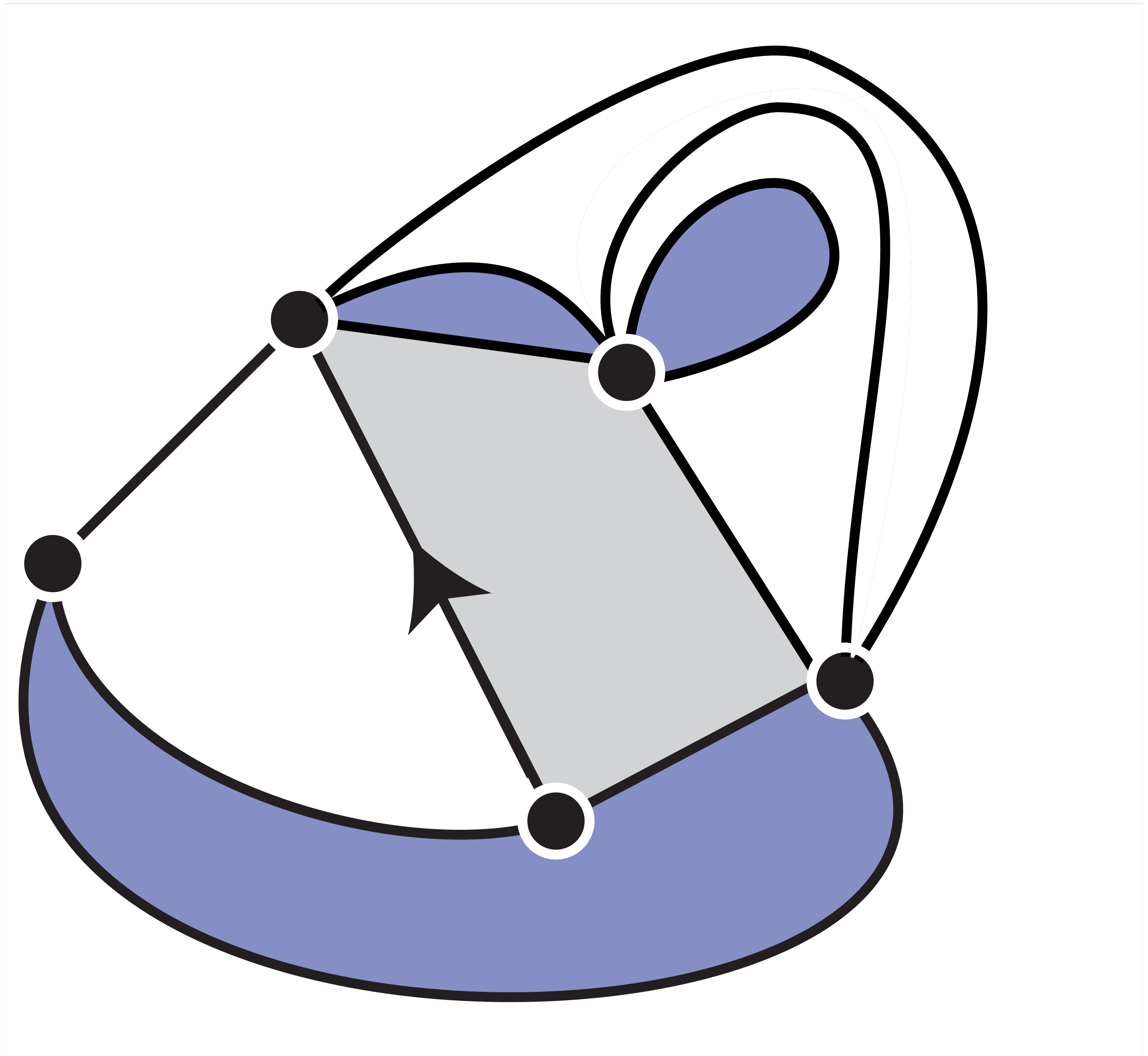

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$\rightsquigarrow$ Key point: $(\widetilde{L}^{(p)}(i); i \geqslant 0)$ is a Markov chain starting at p, absorbed at 0 and with explicit transitions. In addition, the triangulations filling-in the holes of non-explored regions are independent Boltzmann triangulations with a boundary.

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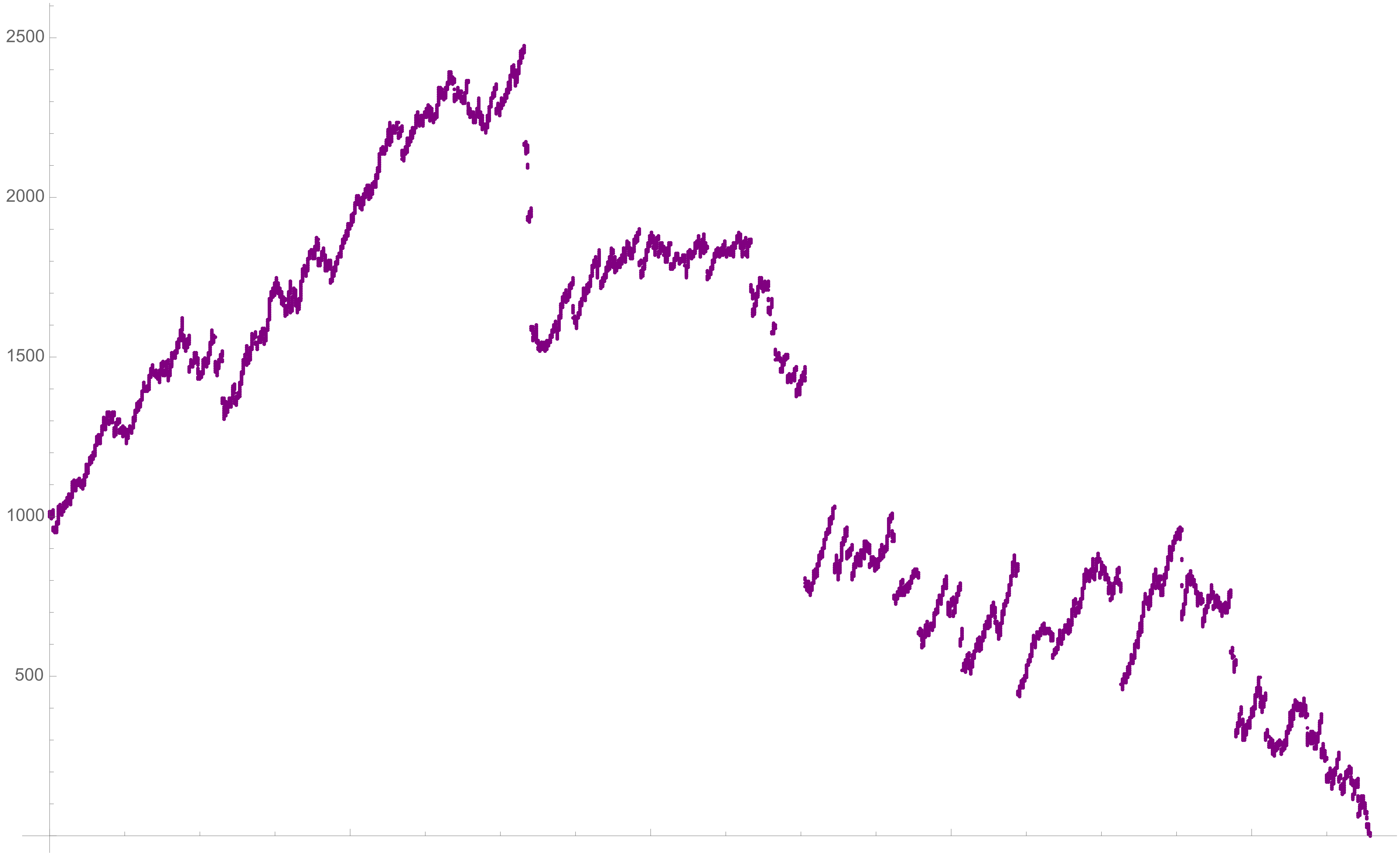
A simulation of X



The self-similar Markov process X

Let ξ be a spectrally negative Lévy process with Laplace exponent

$$\Psi(q) = -\frac{8}{3}q + \int_{1/2}^{1} (x^q - 1 + q(1-x)) \left(x(1-x)\right)^{-5/2} dx,$$

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Description of the limiting process: a growth-fragmentation process

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By Bertoin '15, for every $t \geqslant 0$, the family of all the cells alive at time t is cube summable, and can thus be rearranged in decreasing order. This yields a random variable with values in $\ell_3^{\downarrow}$ denoted by $\mathbb{X}(t) = (X_1(t), X_2(t), \ldots)$.

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- We have $\int^1 (1-x)^2\nu(dx) < \infty$, but $\int^1 (1-x)\nu(dx) = \infty$ which underlines the necessity of compensating the dislocations.

An artistic representation of a growth-fragmentation

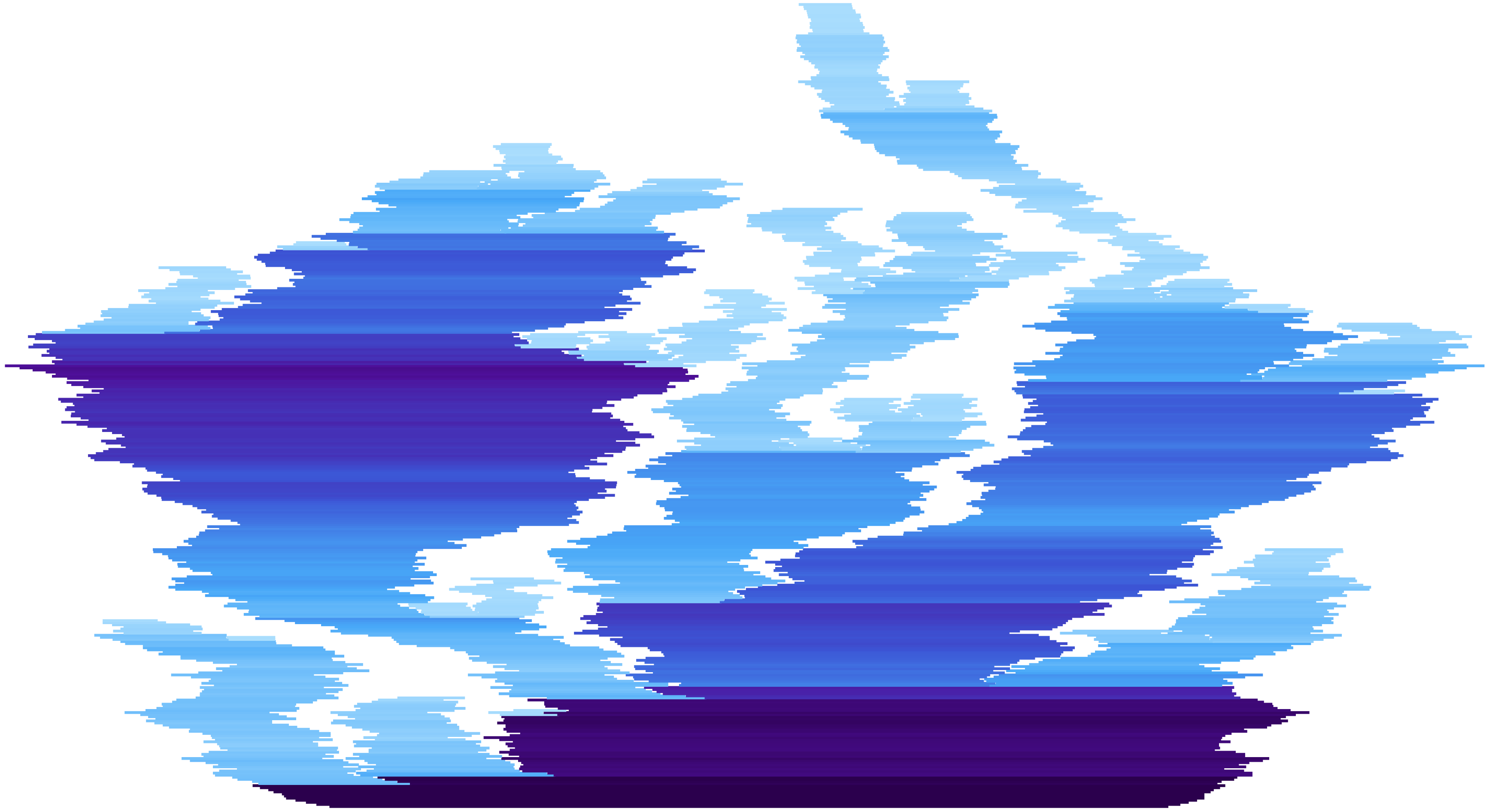

Figure: An artistic representation (by N. Curien) of the cycle lengths at fixed heights of a Boltzmann triangulation with a large boundary: horizontal segments correspond to cycle lengths (the darker the cycle is, the longer it is).

The theorem

Recall that $\mathbb{L}^{(p)}(r) = \left(L_1^{(p)}(r), L_2^{(p)}(r), \ldots\right)$ are the lengths of the cycles of $B_r(T^{(p)})$ ranked in decreasing order.

Theorem (Bertoin, Curien, K. '15)**.**

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in distribution in $\ell_3^{\downarrow}$, where $\mathbb{X} = (\mathbb{X}(t); t \geqslant 0)$ is a càdlàg process with values in $\ell_3^{\downarrow}$, which is a *self-similar growth-fragmentation process* (Bertoin '15).

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$$\kappa(q) = \frac{4\sqrt{\pi}}{3} \frac{\Gamma(q - \frac{3}{2})}{\Gamma(q-3)}, \qquad q > 3/2.$$

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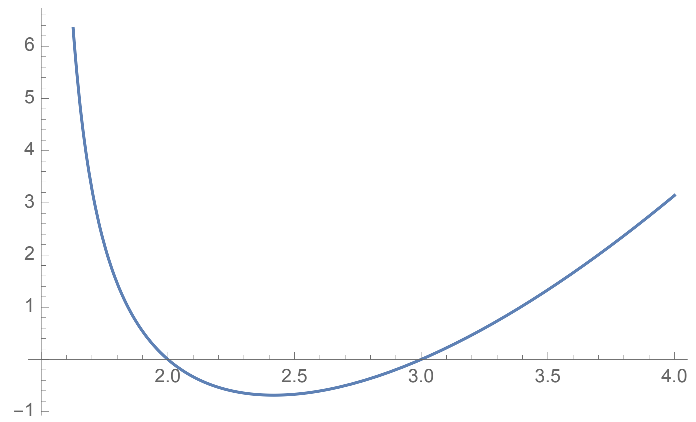

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⇝ Conversely, if one assumes that the evolution of the tagged cell when biasing with the martingale associated with ω_- is a spectrally negative α-stable process conditioned to die at 0 continuously, then $\alpha = 3/2$ and $\kappa(q) = \frac{4\sqrt{\pi}}{3} \frac{\Gamma(q-\frac{3}{2})}{\Gamma(q-3)}$, $q > 3/2$ (use Kuznetsov & Pardo).

Extension to other models of planar maps

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Question. Find the asymptotic behavior of the tail of the extinction time of these growth-fragmentations.