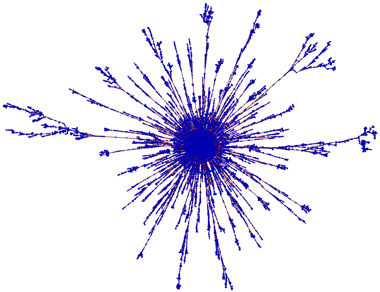
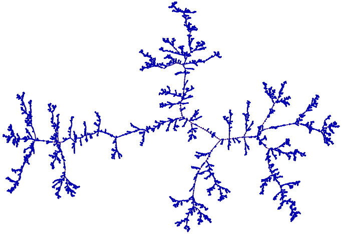
Limit theorems for conditioned non-generic Galton-Watson trees





Igor Kortchemski (Université Paris-Sud, Orsay)
PIMS 2012

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Two approaches:

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What happens when μ is not critical?

Outline

I. State of the art (critical case)

II. Non-generic trees

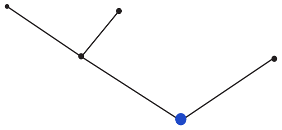
III. Limit theorems for non-generic trees

IV. One conjecture and one problem

I. State of the art

Recap on Galton-Watson trees

Trees will be planar and rooted.



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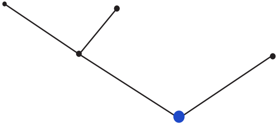

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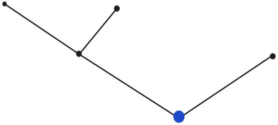

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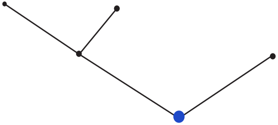

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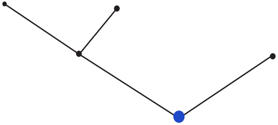

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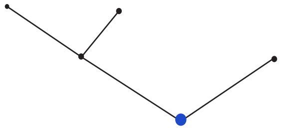

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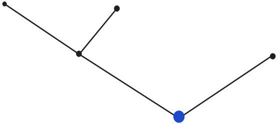

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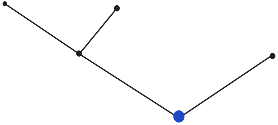

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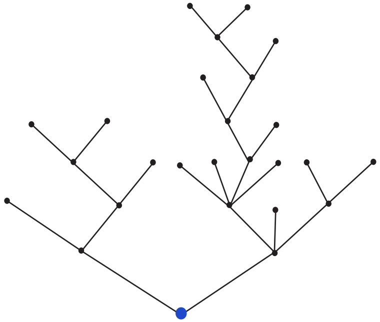
Coding trees



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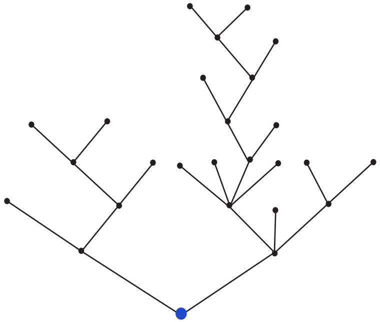

Order the vertices in the ***the lexicographical order***:
$k_\varnothing = u(0) < u(1) < \cdots < u(\zeta(\tau) - 1)$.

Let k_u be the number of children of the vertex u.

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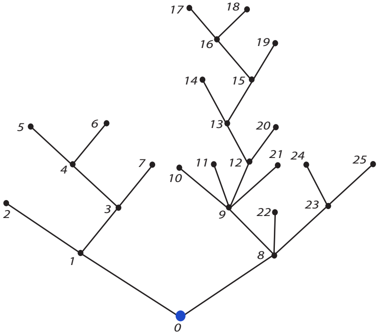

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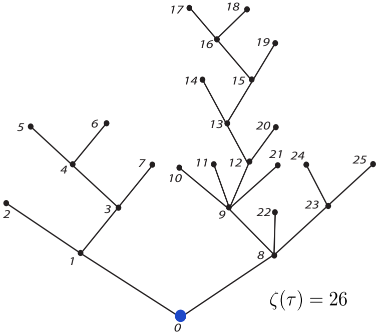

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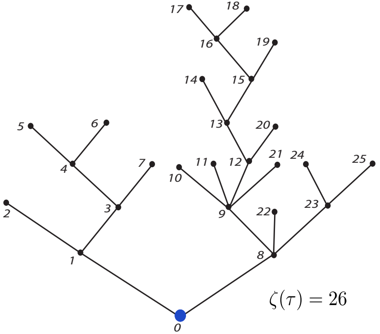
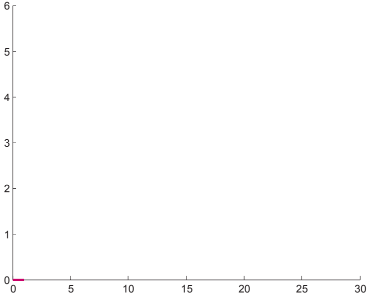


$\zeta(\tau) = 26$



Definition

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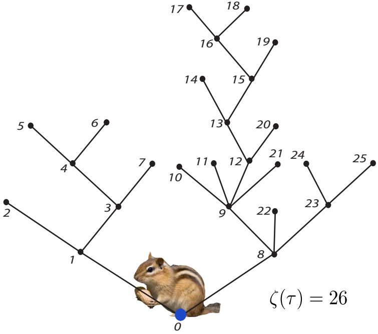

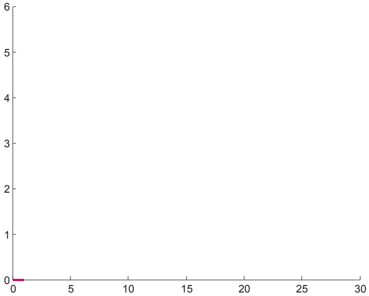

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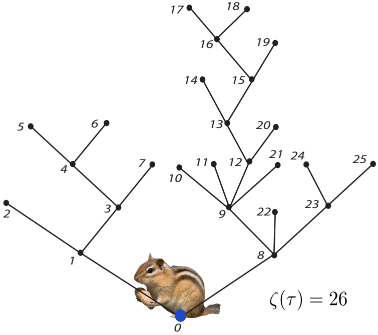

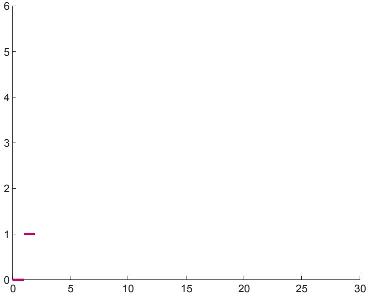

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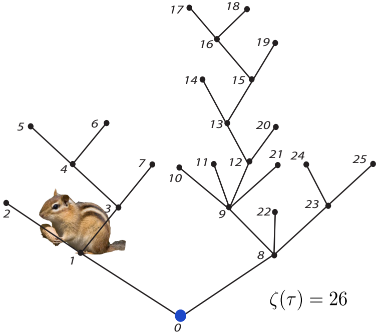

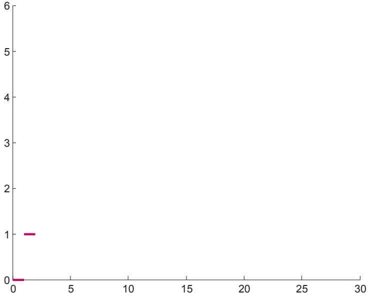

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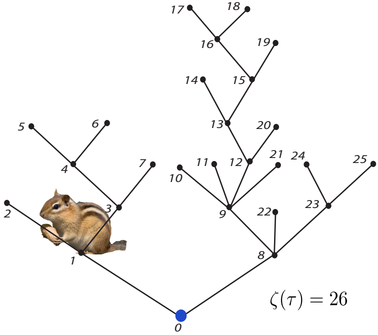

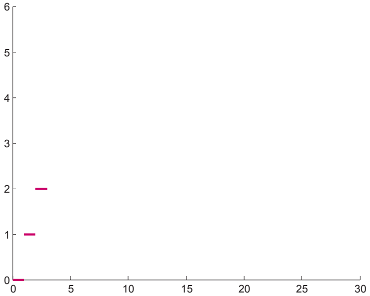

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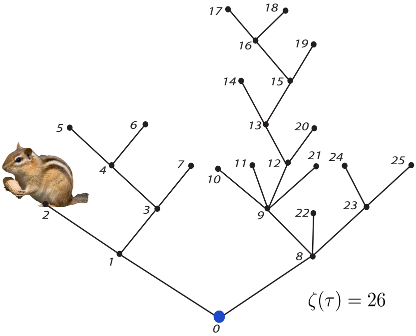

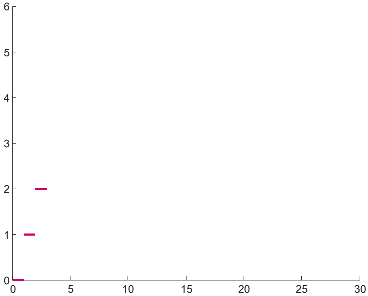

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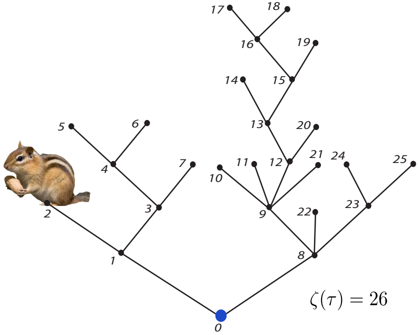

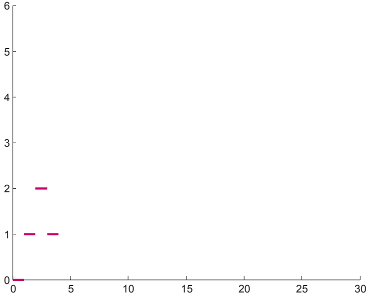

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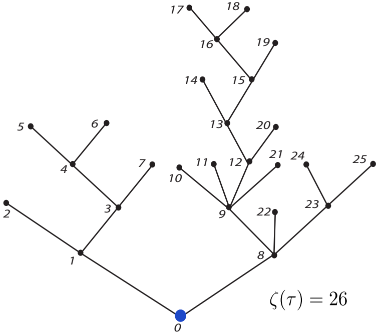

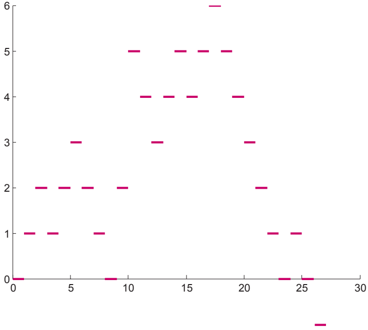

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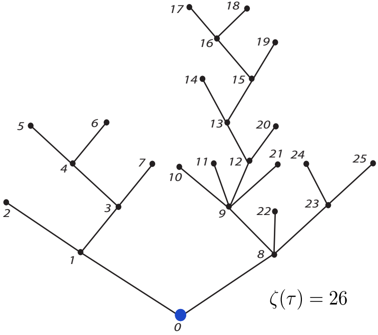
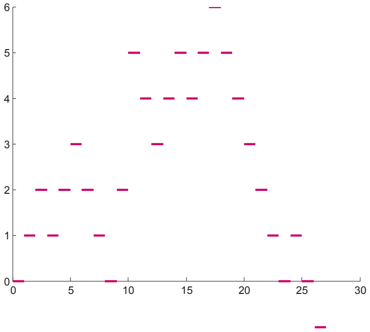
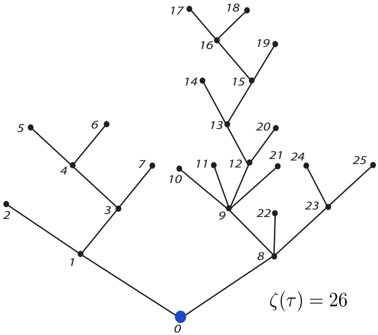




Proposition

The Lukasiewicz path of a GW_μ tree has the same distribution as a random walk with jump distribution $\nu(k) = \mu(k+1)$, $k \geqslant -1$, started from 0, stopped when it hits -1.

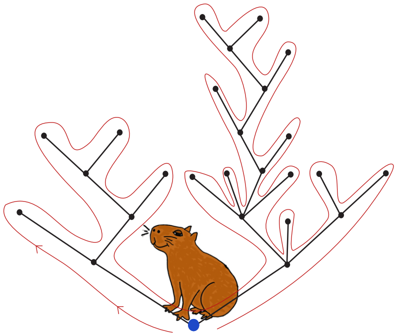
Coding trees



Definition (of the contour function)

A capybara explores the tree at unit speed. For $0 \leqslant t \leqslant 2(\zeta(\tau) - 1)$, $C_t(\tau)$ is the distance between the beast at time t and the root.

Coding trees



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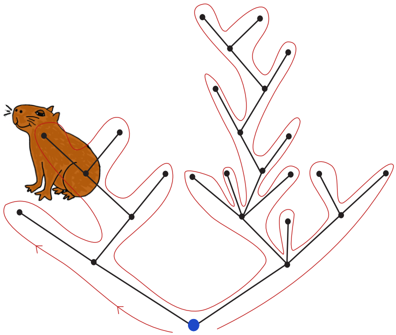

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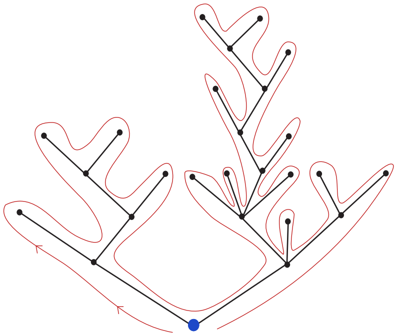

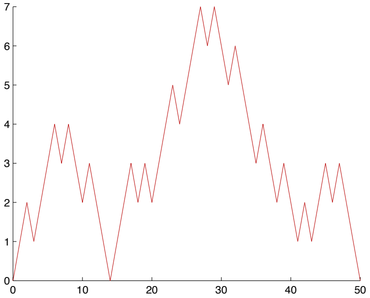

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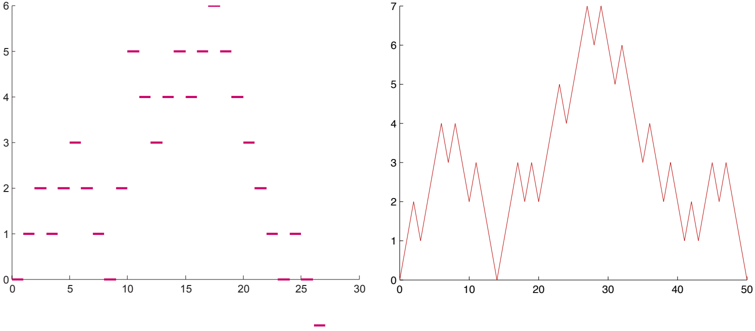

Figure: The Lukasiewicz path and the contour function.

- The Lukasiewicz path behaves like a random walk.

Scaling limits

Let μ be a critical offspring distribution with finite variance. Let $\mathfrak{t}_n$ be a $\mathbb{P}_\mu[\cdot \mid \zeta(\tau) = n]$ tree. What does $\mathfrak{t}_n$ look like for n large ?

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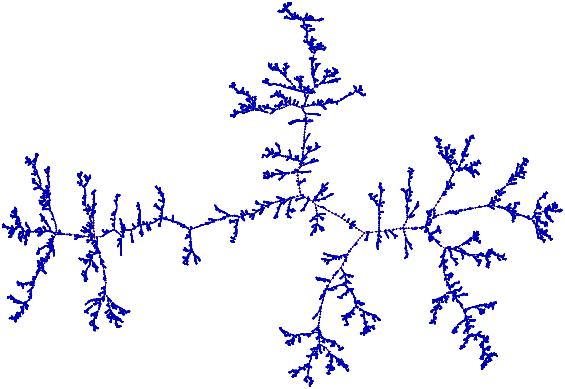

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Theorem (Aldous '93, Duquesne '04)

Let σ^2 be the variance of μ. Then :

$$\left(\frac{1}{\sqrt{n}} \mathcal{W}_{[nt]}(\mathfrak{t}_n), \frac{1}{2\sqrt{n}} C_{2nt}(\mathfrak{t}_n) \right)_{0 \leqslant t \leqslant 1} \quad \xrightarrow[n \to \infty]{(d)} \quad \left(\sigma \cdot \mathbb{e}(t), \frac{1}{\sigma} \mathbb{e}(t) \right)_{0 \leqslant t \leqslant 1},$$

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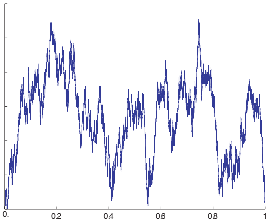

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Remark:

- Duquesne '04: extension to the case where μ is in the domain of attraction of a stable law.

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where $\mathbb{e}$ is the normalized Brownian excursion.

Consequences:

- limit theorem for the height of t_n,

Scaling limits

Let μ be a critical offspring distribution with finite variance. Let $\mathfrak{t}_n$ be a $\mathbb{P}_\mu[\cdot \mid \zeta(\tau) = n]$ tree. What does $\mathfrak{t}_n$ look like for n large ?

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where $\mathbb{e}$ is the normalized Brownian excursion.

Consequences:

- limit theorem for the height of $\mathfrak{t}_n$,
- convergence in the Gromov-Hausdorff sense of $\mathfrak{t}_n$, suitably rescaled, towards the Brownian CRT.

II. Non-generic trees

II. 1) Exponential families

Exponential families

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Then a GW_μ tree conditioned on having n vertices has the same distribution as a $\mathrm{GW}_{\mu^{(\lambda)}}$ tree conditioned on having n vertices.

Consequence:

- if there exists $\lambda > 0$ such that $Z_\lambda < \infty$ and $\mu^{(\lambda)}$ is critical, then we are back to the critical case.

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Definition

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Example: $\mu(i) \sim c/i^\beta$ with $c > 0$ and $\beta > 2$.

II. 2) Large non-generic trees

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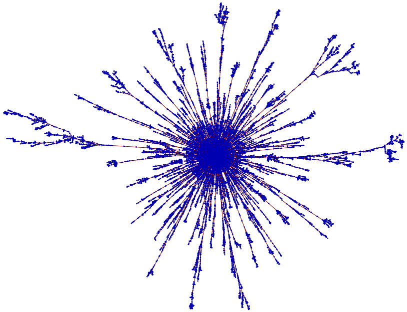

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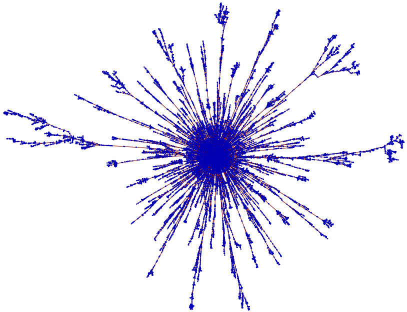

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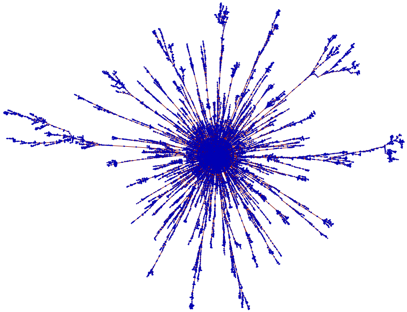

Condensation phenomenon (which also appears in the zero-range process !).

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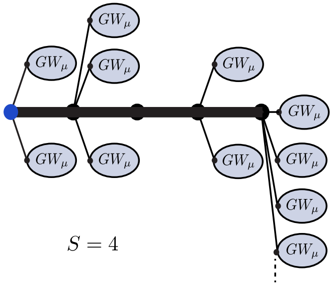

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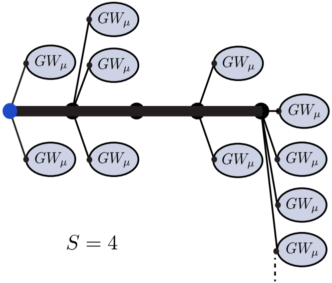

The spine has a finite random length S, *where:*
$\mathbb{P}[S = i] = (1 - \mathbf{m})\mathbf{m}^i$ *for* $i \geqslant 0$

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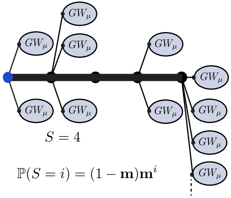

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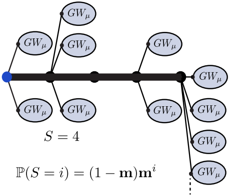

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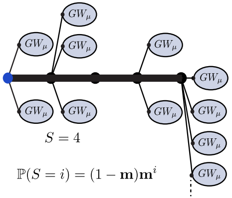

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- In the critical case the spine is infinite (Kesten '86).

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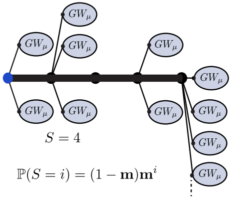

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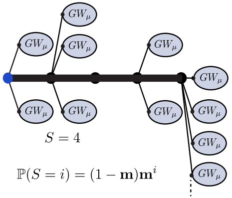

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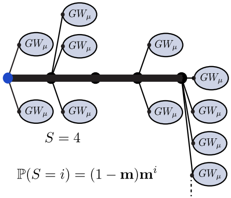

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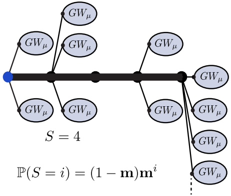

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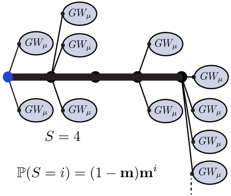

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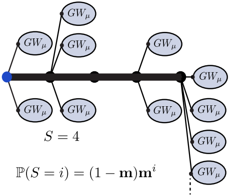

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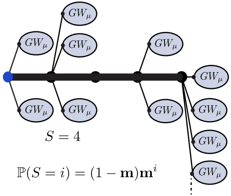

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- What are the fluctuations of the maximal degree?
- Where is located the vertex of maximal degree ?
- What is the height of t_n?

III. Limit theorems for non-generic trees

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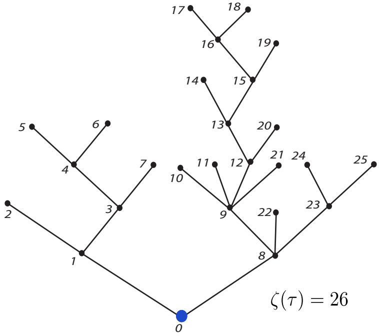
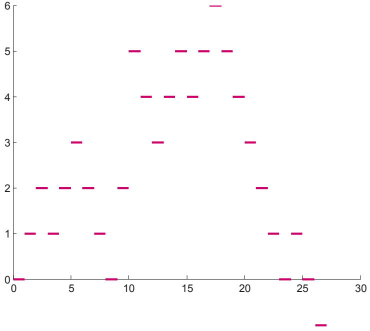
III. 1) Convergence of the Lukasiewicz path

Recap





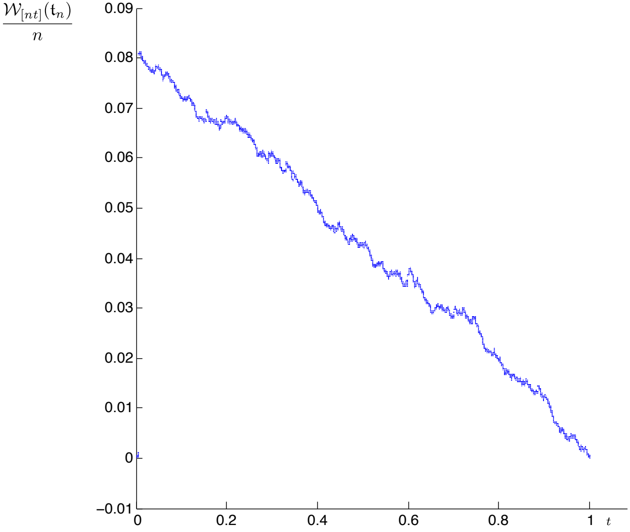
Let k_u be the number of children of the vertex u.

Definition

The Lukasiewicz path $\mathcal{W}(\tau) = (\mathcal{W}_n(\tau), 0 \leqslant n \leqslant \zeta(\tau))$ of a tree τ is defined by :

$$\mathcal{W}_0(\tau) = 0, \qquad \mathcal{W}_{n+1}(\tau) = \mathcal{W}_n(\tau) + k_{u(n)}(\tau) - 1.$$

Convergence of the Lukasiewicz path



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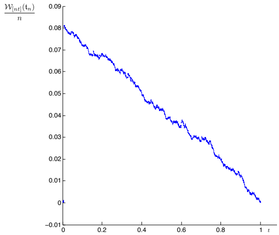
Let $U(\mathfrak{t}_n)$ be the index of the first vertex with maximal degree of $\mathfrak{t}_n$.



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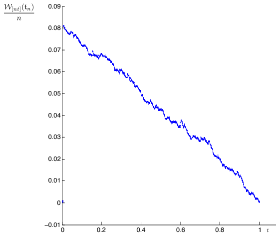

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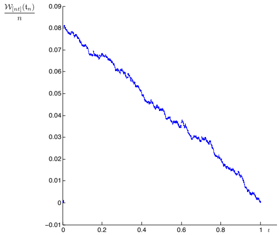

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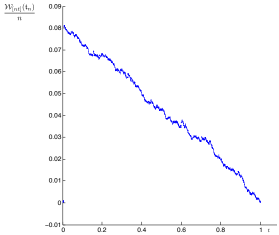

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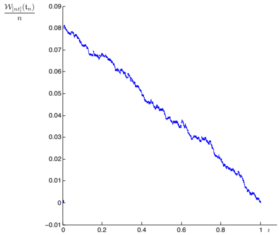
(iii) $$\left(\frac{\mathcal{W}_{[nt] \vee (U(t_n)+1)}(t_n)}{n}, 0 \leqslant t \leqslant 1 \right) \quad \xrightarrow[n\to\infty]{(d)} \quad ((1-\mathbf{m})(1-t))_{0 \leqslant t \leqslant 1}.$$



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Remarks:

- The limit is deterministic and depends only on $\mathbf{m}$ (the mean of μ).

Convergence of the Lukasiewicz path

Let $U(t_n)$ be the index of the first vertex with maximal degree of t_n.

Theorem (K. 12')

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- We know that $\mathcal{W}(\mathfrak{t}_n)$ has the law of a random walk $(W_n)_{n \geqslant 0}$ with jump distribution $\nu(k) = \mu(k+1)$, $k \geqslant -1$

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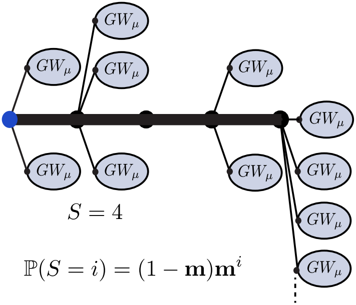

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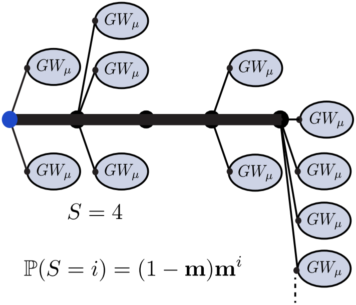

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- For every sequence $(\lambda_n)_{n\geqslant 1}$ such that $\lambda_n \to +\infty$:

$$\mathbb{P}\left[\left|\mathcal{H}(t_n) - \frac{\ln(n)}{\ln(1/\mathbf{m})}\right| \leqslant \lambda_n\right] \underset{n\to\infty}{\longrightarrow} 1.$$

IV. Extensions

Conjecture

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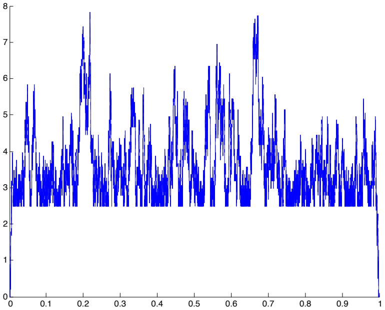
Contour function of $\mathfrak{t}_n$



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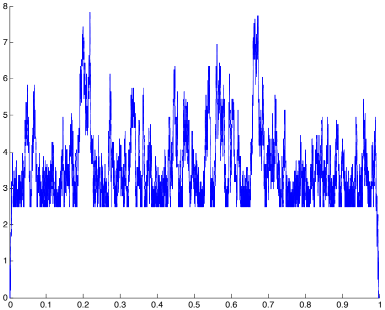
Theorem (K. 12')

Let $(r_n)_{n \geqslant 1}$ be a sequence of positive real numbers.



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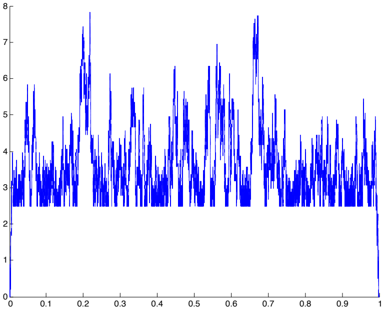
Let $(r_n)_{n \geqslant 1}$ *be a sequence of positive real numbers.*

(i) *If* $r_n / \ln(n) \to \infty$, *then* $(C_{2nt}(\mathfrak{t}_n)/r_n, 0 \leqslant t \leqslant 1)$ *converges to the function equal to* 0 *on* $[0,1]$ *as* $n \to \infty$.



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(ii) *Otherwise, the sequence* $(C_{2nt}(\mathfrak{t}_n)/r_n, 0 \leqslant t \leqslant 1)$ *is not tight in the space* $\mathcal{C}([0,1], \mathbb{R})$*.*