Solution to the exercise

(i) We saw that $\mathbb{P}(W_{T_1}=k)=\bar{\mu}(k)=\mu([k+1,\infty))$.

Hence $\mathbb{E}[W_{T_1}]=\sum_{k=1}^{\infty} k\mu([k+1,\infty))=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} k\,\mathbb{1}_{j\geq k+1}\,\mu(j)$

$$=\sum_{j=1}^{\infty}\mu(j)\cdot\sum_{k=1}^{j-1}k=\sum_{j=1}^{\infty}\mu(j)\cdot\frac{j(j-1)}{2}$$

$$=\Big(\sum_{j=1}^{\infty}\mu(j)j^2-\sum_{j=1}^{\infty}j\mu(j)\Big)/2$$

$$=\sigma^2/2.$$

(ii) We introduce $M_n=\max_{0\leq i\leq n} W_i$ and recall that

$\widehat{W}^{(n)}=(W_n-W_{n-i};0\leq i\leq n)$.

We saw that $H_n=\left|\{1\leq k\leq n;\ \widehat{W}_k^{(n)}=\max_{0\leq j\leq k}\widehat{W}_j^{(n)}\}\right|$.

Also, $W_n-I_n=W_n-\min_{0\leq i\leq n}W_i=\max_{0\leq i\leq n}(W_n-W_i)$

$=\max_{0\leq i\leq n}\widehat{W}_i^{(n)}$.

It follows that $(H_n,W_n-I_n)\overset{(d)}{=}(R_n,M_n)$.

It is therefore enough to show that $\dfrac{R_n}{M_n}\xrightarrow[n\to\infty]{(\mathbb{P})}\dfrac{2}{\sigma^2}$ $(*)$

To this end, note that

$$M_n=\sum_{k;\,T_k\leq n}(W_{T_k}-W_{T_{k-1}})=\sum_{k=1}^{R_n}(S_{T_k}-S_{T_{k-1}})$$

Since $(S_{T_k}-S_{T_{k-1}};k\geq1)$ are iid of expectation $\sigma^2/2$,

we have $\dfrac{\sum_{k=1}^{m}(S_{T_k}-S_{T_{k-1}})}{m}\xrightarrow{a.s.}\sigma^2/2$.

But since (W_n) is recurrent $R_n\xrightarrow[n\to\infty]{a.s.}\infty$. Hence

$\dfrac{M_n}{R_n}\xrightarrow[n\to\infty]{a.s}\dfrac{\sigma^2}{2}$. This implies $(*)$