Also, $\left(\frac{S_{nt}}{\sigma\sqrt{n}} - \inf_{0\le s\le t} \frac{S_{ns}}{\sigma\sqrt{n}} ; 0\le t\le 1\right) \xrightarrow[n\to\infty]{(d)} \left(B_t - \inf_{0\le s\le t} B_s ; 0\le t\le 1\right)$

and $\frac{S_n - I_n}{\sigma\sqrt{n}} \xrightarrow[n\to\infty]{(d)} B_1 - \inf_{0\le s\le 1} B_s$,

where $I_n = \inf_{0\le i\le n} S_i$.

5) Functional invariance principle for the height function.

Recall the previous notation:

- μ is a critical offspring distribution on $\mathbb{Z}_+$, with finite positive variance σ^2
- $(W_n)_{n\geq 0}$ is a random walk with $\mathbb{P}(W_1 = i) = \mu(i+1)$
- $H_n = \left|\left\{ 0\le i\le n-1 ; W_i = \min_{i\le j\le n} W_j \right\}\right|$ is the height function
- $R_n = \left|\left\{ 1\le k\le n ; W_k = \max_{0\le j\le k} W_j \right\}\right|$ is the number of weak records between times 1 and n
- $M_n = \max_{0\le i\le n} W_i$
- $I_n = \min_{0\le i\le n} W_i$

We saw that $(M_n, R_n) \overset{(d)}{=} (W_n - I_n, H_n)$, that $\frac{H_n}{W_n - I_n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{2}{\sigma^2}$

and that $\left(\frac{W_{nt} - I_{nt}}{\sigma\sqrt{n}} ; 0\le t\le 1\right) \xrightarrow[n\to\infty]{(d)} \left(B_t - \inf_{0\le s\le t} B_s ; 0\le t\le 1\right)$.

As a consequence, $\boxed{\frac{H_n}{\sqrt{n}} \xrightarrow[n\to\infty]{(d)} \frac{2}{\sigma}\left(B_1 - \inf_{0\le s\le 1} B_s\right)}$

We show that actually more is true:

Thm Assume that in addition $\exists \lambda > 0$ s.t. $\sum_{k=0}^{\infty} e^{\lambda k}\mu(k) < \infty$.

Then $\left(\frac{1}{\sqrt{n}} H_{nt} ; 0\le t\le 1\right) \xrightarrow[n\to\infty]{(d)} \left(\frac{2}{\sigma}\left(B_t - \inf_{0\le s\le t} B_s\right) ; 0\le t\le 1\right)$

The idea is to show that

$$\sup_{0\le t\le 1} \left| \frac{W_{nt} - I_{nt}}{\sqrt{n}} - \frac{\sigma^2}{2}\frac{H_{nt}}{\sqrt{n}} \right| \xrightarrow[n\to\infty]{a.s.} 0.$$

or, equivalently, that $\sup_{0\le t\le 1} \left| \frac{M_{nt}}{\sqrt{n}} - \frac{\sigma^2}{2}\frac{R_{nt}}{\sqrt{n}} \right| \xrightarrow[n\to\infty]{a.s} 0$ $(*)$

This indeed implies the theorem, since if $X_n \xrightarrow{(d)} X$ and $Y_n \xrightarrow{a.s} 0$ then $X_n + Y_n \xrightarrow{(d)} X$.

Notation: If (a_n) is a sequence of real numbers, we write $a_n = oe(n)$ if there exists $\varepsilon>0$ s.t $|a_n| \le e^{-n^{\varepsilon}}$ for every n large enough.

We shall show:

Lemma 1 We have $\sup_{0\le k\le n} \mathbb{P}\left(\left| M_k - \frac{\sigma^2}{2} R_k \right| > n^{3/8} \right) = oe(n)$.

Lemma 1 implies $(*)$. Indeed,

$$\mathbb{P}\left(\sup_{0\le t\le 1} \left| \frac{M_{nt}}{\sqrt{n}} - \frac{\sigma^2}{2}\frac{R_{nt}}{\sqrt{n}} \right| > \frac{1}{n^{1/8}} \right)$$
$$\le \mathbb{P}\left(\exists\, 1\le k\le n \,;\, \left| M_k - \frac{\sigma^2}{2} R_k \right| > n^{3/8} \right)$$
$$\le n \sup_{1\le k\le n} \mathbb{P}\left(\left| M_k - \frac{\sigma^2}{2} R_n \right| > n^{3/8} \right) = oe(n)$$

Hence $\sum_{n\ge 1} \mathbb{P}\left(\sup_{0\le t\le 1} \left| \frac{M_{nt}}{\sqrt{n}} - \frac{\sigma^2}{2}\frac{R_{nt}}{\sqrt{n}} \right| > \frac{1}{n^{1/8}} \right) < \infty$.

Therefore, by Borel-Cantelli's lemma, a.s, $\exists N$ (random) s.t $n\ge N \Rightarrow \sup_{0\le t\le 1} \left| \frac{M_{nt}}{\sqrt{n}} - \frac{\sigma^2}{2}\frac{R_{nt}}{\sqrt{n}} \right| \le \frac{1}{n^{1/8}}$

This implies $(*)$.

To show Lemma 1, we will use:

Lemma 2 Let $(Y_i)_{i\ge 1}$ be a sequence of iid random variables such that $\exists \lambda>0$ s.t $\mathbb{E}[e^{\lambda |Y_1|}] < \infty$ and $\mathbb{E}[Y_1]=0$. Then for every $\varepsilon>0$, $\mathbb{P}\left(\sup_{0\le k\le n} |Y_1 + \dots + Y_k| > n^{\frac{1}{2}+\varepsilon} \right) = oe(n)$.

Proof of Lemma 2: The assumption implies $\mathbb{E}[e^{\lambda Y_1}] = 1 + c\lambda^2 + O(\lambda^3)$ as $\lambda \to 0$, with $c = \mathrm{Var}(Y_1)/2$. Hence $\exists C>0$ s.t $\mathbb{E}[e^{\lambda Y_1}] \le e^{C\lambda^2}$ for λ small enough.

Hence, for λ small enough, and $1 \leq k \leq n$,

$$\mathbb{P}(Y_1 + \cdots + Y_k > n^{\frac{1}{2}+\alpha}) \leq e^{-\lambda n^{1/2+\varepsilon}} \mathbb{E}[e^{\lambda(Y_1 + \cdots + Y_k)}]$$

$$\leq e^{-\lambda n^{\frac{1}{2}+\varepsilon} + Cn\lambda^2}.$$

Take $\lambda = \frac{1}{\sqrt{n}}$, and we get that for n large enough

$$\mathbb{P}(Y_1 + \cdots + Y_k > n^{\frac{1}{2}+\alpha}) \leq e^{C - n^{\varepsilon}}.$$

The result follows (by also replacing Y_i by $-Y_i$ to get $|Y_1 + \cdots + Y_k|$).

$\boxed{\Xi}$

$\boxed{\text{We now show Lemma 1}}$

Recall that $T_0 = 0$ and $T_i = \inf\{n > T_{i-1} ;\ W_n \geq W_{T_{i-1}}\}$ are the weak record times of W, that

$$M_n = \sum_{k=1}^{R_n} (W_{T_k} - W_{T_{k-1}}),$$

that $(W_{T_k} - W_{T_{k-1}} ; k \geq 1)$ are iid with $\mathbb{P}(W_{T_1} = k) = \mu([k+1, \infty[)$.

Now, for $i \geq 1$, set $Y_i = W_{T_i} - W_{T_{i-1}} - \frac{\sigma^2}{2}$, so that $(Y_i)_{i \geq 1}$ satisfies the assumptions of Lemma 1, and

$$\boxed{M_k - \frac{\sigma^2}{2} R_k = \sum_{i=1}^{R_k} Y_i}$$

Now fix $\varepsilon > 0$ small enough so that $\frac{3}{8} > \frac{1}{2}\left(\frac{1}{2} + \varepsilon\right)$, and set $m_n = \lfloor n^{1/2+\varepsilon} \rfloor$, and write

$$\mathbb{P}\left(\sup_{0 \leq k \leq n} |M_k - \frac{\sigma^2}{2} R_k| > n^{3/8}\right) \leq \mathbb{P}(R_n > m_n) + \mathbb{P}\left(\sup_{0 \leq k \leq m_n} \left|\sum_{i=1}^{k} Y_i\right| > n^{3/8}\right)$$

By Lemma 4.1, the second term is $oe(n)$.

For the first term, write

$$\mathbb{P}(R_n \geq m_n) = \mathbb{P}(W_{T_{m_n}} \leq M_n)$$

$$\leq \mathbb{P}(W_{T_{m_n}} \leq n^{\frac{1}{2}+\frac{\varepsilon}{2}}) + \mathbb{P}(M_n \geq n^{\frac{1}{2}+\frac{\varepsilon}{2}})$$

But. $\mathbb{P}(M_n \geq n^{\frac{1}{2}+\frac{\varepsilon}{2}}) = \mathbb{P}(\sup_{0\leq k\leq n} S_k \geq n^{\frac{1}{2}+\frac{\varepsilon}{2}})$
$= o_e(n)$ by Lemma 4.1

and

$$\mathbb{P}(W_{T_{m_n}} \leq n^{\frac{1}{2}+\frac{\varepsilon}{2}}) = \mathbb{P}(W_{T_{m_n}} - \frac{\sigma^2}{2} m_n \leq n^{\frac{1}{2}+\frac{\varepsilon}{2}} - \frac{\sigma^2}{2} m_n)$$

But $W_{T_{m_n}} - \frac{\sigma^2}{2} m_n$ is the sum of m_n iid random variables satisfying the assumption of Lemma 4.1, and $n^{\frac{1}{2}+\frac{\varepsilon}{2}} = o(m_n)$.

Hence $\mathbb{P}(W_{T_{m_n}} \leq n^{\frac{1}{2}+\frac{\varepsilon}{2}}) = o_e(n)$. $\blacksquare$

NB It also possible to show that the theorem holds more generally if only $\sigma^2 < \infty$.

VII Scaling limits of large conditioned GW trees and the Brownian excursion.

Here μ is a critical offspring distribution on $\mathbb{Z}_+$ s.t $\exists \lambda > 0$ s.t $\sum_{k\geq 1} e^{\lambda k} \mu(k) < \infty$

Let σ^2 be the variance of μ.

Recall that $\mathbb{P}_\mu$ is the law of a GW_μ.

Assume that $\sigma^2 > 0$ and that μ is aperiodic (so that $\mathbb{P}_\mu(|T| = n) > 0$ for large n).

Let T_n be a GW_μ tree conditioned on having n vertices.

Our goal is to show the following result

Theorem Let $(H_k(T_n)\,;\, 0 \leq k \leq n-1)$ be the height function of T_n. Set $H_n(T_n) = 0$. There exists a random element of E denoted by $\mathbb{e}$ such that

$$\left(\frac{H_{nt}(T_n)}{\sqrt{n}}\,;\, 0 \leq t \leq 1\right) \xrightarrow[n\to\infty]{(d)} \mathbb{e}.$$

$\mathbb{e}$ is called the Brownian excursion

We will first construct $\mathbb{P}$

1) The Brownian Bridge

Proposition As $\varepsilon \downarrow 0$, the law of $(B_s\,;\,0 \leq s \leq 1)$ conditionally on $\{|B_1| \leq \varepsilon\}$ converges weakly to a probability distribution on E denoted by $\mathbb{P}^0$. In addition, $\mathbb{P}^0$ is the law of

$$(B_s - sB_1\,;\,0 \leq s \leq 1).$$

Proof: set $b_s = B_s - sB_1$ for $0 \leq s \leq 1$.

Step 1 For $s \in [0,1]$, $\mathrm{Cov}(b_s, B_1) = \mathbb{E}[(B_s - sB_1)B_1]$

$$= \mathbb{E}[B_s B_1] - s$$

$$= \mathbb{E}[B_s(B_1 - B_s) + B_s^2] - s$$

$$= \mathbb{E}[B_s]\,\mathbb{E}[B_1 - B_s] + 0 \quad \text{since } B_s \perp\!\!\!\perp B_1 - B_s$$

$$= 0.$$

This implies that $b \perp\!\!\!\perp B_1$ (exercise)