Recall μ is offspring distribution

Extension to forests

Denote by $\mathbb{P}_{\mu,j}$ the probability measure on $\mathbb{T}_f^j$ which is the law of the forest of j independent GW_μ trees.

Denote by $(\widetilde{W}_0(f), \ldots, \widetilde{W}_{|f|}(f))$ its Lukasiewicz path.

Then under $\mathbb{P}_{\mu,j}$,
$$\left(\widetilde{W}_0(f), \ldots, \widetilde{W}_{|f|}(f)\right) \overset{(d)}{=} (W_0, W_1, \ldots, W_{\zeta_k})$$
where $\zeta_k = \inf\{k \geq 1 ;\ W_k = -1\}$.

In addition,
$$\boxed{\mathbb{P}_{\mu,j}(|F| = n) = \frac{j}{n}\,\mathbb{P}(W_n = -1)}$$

We would like therefore to estimate $\mathbb{P}(W_n = -1)$!.

Crucial point
$$\mathbb{E}[W_1] = \sum_{i \geq -1} i\,\mu(i+1) = \sum_{i \geq 0} (i-1)\,\mu(i) = \sum_{i \geq 0} i\,\mu(i) - 1$$

Hence $\boxed{\mathbb{E}[W_1] = 0 \iff \mu \text{ is critical.}}$

5) The local limit theorem

Let $(Z_n)_{n \geq 0}$ be any random walk on $\mathbb{Z}$.

Let $h \in \mathbb{Z}$ be the maximal integer such that Z_1 only takes values in $c + h\mathbb{Z}$.

Then Z_n only takes values in $cn + h\mathbb{Z}$.

Hence, up to considering $\dfrac{Z_n - cn}{h}$, we suppose $\boxed{\begin{matrix} c = 0 \\ h = 1 \end{matrix}}$ [Aperiodic case]

Thm (Local Limit Theorem)

Assume that $\mathbb{E}[|Z_1|] < \infty$ and $\mathbb{E}[Z_1^2] < \infty$.

Set $a = \mathbb{E}[Z_1]$ and $\sigma^2 = \mathbb{E}[Z_1^2] - \mathbb{E}[Z_1]^2$. Then
$$\sup_{k \in \mathbb{Z}} \left| \sigma\sqrt{n}\,\mathbb{P}(Z_n = k) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{k - an}{\sigma\sqrt{n}}\right)^2\right) \right| \xrightarrow[n \to \infty]{} 0$$

The local limit theorem implies the central limit theorem.

For example, for $a=0$:

$$\mathbb{P}\left(u < \frac{Z_n}{\sigma\sqrt{n}} < v\right) \approx \sum_{k=u\sigma\sqrt{n}}^{v\sigma\sqrt{n}} \mathbb{P}(Z_n=k)$$

$$\approx \int_{u\sigma\sqrt{n}}^{v\sigma\sqrt{n}} \mathbb{P}(Z_n=\lfloor x\rfloor)\,dx$$

$$\approx \int_a^b \sigma\sqrt{n}\, \mathbb{P}(Z_n=\lfloor x\sigma\sqrt{n}\rfloor)\,dx$$

$$\approx \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx.$$

Before starting the proof, we prove the following useful result.

Lemma Set $\varphi(t)=\mathbb{E}[e^{itZ_1}]$. Recall that $h=1$.
Then $|\varphi(t)|<1$ for $t\in(0,2\pi)$

<u>Proof</u>: Assume that $|\varphi(t_0)|=1$ for $t_0\in(0,2\pi)$.

Then write $\mathbb{E}[e^{it_0Z_1}] = e^{it_0 a}$ for a certain $a\in\mathbb{R}$.

Hence $\mathbb{E}[e^{it_0(Z_1-a)}]=1$.

Hence $\sum_{k\in\mathbb{Z}} \mathbb{P}(Z_1=k)\cos(t_0(k-a))=1$

Hence $\cos(t_0(k-a))=1$ for every $k\in\mathbb{Z}$ such that $\mathbb{P}(Z_1=k)>0$.

Hence $\text{Support}(Z_1)\subset a+\frac{2\pi}{t_0}\mathbb{Z}$.

by maximality of h, $\frac{2\pi}{t_0}\leq 1$. Contradiction. ↯

We now prove the Local Limit Theorem

Instead of considering Z_n-an, we suppose without loss of generality that $a=0$ $(=\mathbb{E}[Z_1])$

Write $\mathbb{E}[e^{itZ_n}] = \sum_{k\in\mathbb{Z}} e^{itk}\,\mathbb{P}(Z_n=k)$.

Hence $$\mathbb{P}(Z_n=k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itk}\,\mathbb{E}[e^{itZ_n}]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itk}\,\varphi(t)^n\,dt$$

Therefore:

$$\sigma\sqrt{n}\,\mathbb{P}(Z_n = u\sigma\sqrt{n}) = \frac{\sigma\sqrt{n}}{2\pi}\int_{-\pi}^{\pi} e^{-itu\sigma\sqrt{n}}\varphi(t)^n\,dt \quad \left[\begin{array}{c}\text{take}\\ k = u\sigma\sqrt{n}\end{array}\right]$$

$$= \frac{1}{2\pi}\int_{-\pi\sigma\sqrt{n}}^{\pi\sigma\sqrt{n}} e^{-itu}\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)^n dt \quad \left[\begin{array}{c}\text{change of}\\ \text{variables}\\ t \rightsquigarrow \frac{t}{\sigma\sqrt{n}}\end{array}\right]$$

But $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2} = \frac{1}{2\pi}\int_{\mathbb{R}} dt\, e^{-itu - t^2/2}$.

Hence, if one fixes $A, \varepsilon > 0$:

$$\left|\sigma\sqrt{n}\,\mathbb{P}(Z_n = u\sigma\sqrt{n}) - \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\right| \leq \frac{|I_1| + |I_2| + |I_3| + |I_4|}{2\pi} \text{ where}$$

$$\boxed{I_1 = \int_{-A}^{A} e^{-itu}\left(\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)^n - e^{-t^2/2}\right)dt}$$

$$\boxed{I_2 = \int_{A<|t|<\varepsilon\sigma\sqrt{n}} e^{-itu}\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)^n dt}$$

$$\boxed{I_3 = \int_{\varepsilon\sigma\sqrt{n}<|t|<\pi\sigma\sqrt{n}} e^{-itu}\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)^n dt}$$

$$\boxed{I_4 = \int_{|t|>A} e^{-itu-t^2/2}\,dt}$$

Fix $\varepsilon' > 0$. We show that we can choose $\left\{\begin{array}{l}\varepsilon>0\\ A>0\end{array}\right.$ such that for every n sufficiently large, $|I_1| + |I_2| + |I_3| + |I_4| \leq 4\varepsilon'$ for every $u \in \mathbb{R}$.

We now control each one of the four terms.

For I_1: we have $I_1 = \int_{-A}^{A} f_n(u,t)\,dt$

with $|f_n(u,t)| \leq 1 + e^{-t^2/2}$ and $\sup_{u\in\mathbb{R}} f_n(u,t) \xrightarrow[n\to\infty]{} 0$ by the central limit theorem.

Hence $I_1 \xrightarrow[n\to\infty]{} 0$ uniformly in u.

For I_4: we have $|I_4| \leq 2\int_A^\infty e^{-t^2/2}\,dt$ which is $\leq \varepsilon'$ provided that A is sufficiently large.

For I_3 By the Lemma, $|\varphi(t)|<1$ for $t \in (0,2\pi)$.
Hence $\exists c>0$ s.t $\left|\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)\right| \leq e^{-c}$ for $\varepsilon\sigma\sqrt{n} < |t| < \pi\sigma\sqrt{n}$.
therefore

$$|I_3| \leq 2\int_{\varepsilon\sigma\sqrt{n}}^{\pi\sigma\sqrt{n}} e^{-cn}\,dt \leq 2\pi\sigma\sqrt{n}\,e^{-cn},$$

which is $\leq \varepsilon'$ for every n sufficiently large.

For I_2 Since $\sigma^2<\infty$, we now that

$$\varphi(t) = 1 - \frac{t^2\sigma^2}{2} + o(t^2). \quad (*)$$

In particular, for t sufficiently small, $|\varphi(t)-1| \leq \frac{1}{2}$.
Hence, for t sufficiently small we may write
$\varphi(t) = r(t)e^{i\theta(t)}$ with r, θ continuous,
and $\ln\varphi(t) = \ln r(t) + i\theta(t)$, with $\ln$ analytic in $\{z \in \mathbb{C} ; |z-1| \leq \frac{1}{2}\}$.

In particular $\ln(1+z) \underset{z\to 0}{=} z + o(z)$

Hence $\ln\varphi(t) = -\frac{t^2\sigma^2}{2} + o(t^2)$

Hence $\ln r(t) = \mathrm{Re}(\ln\varphi(t)) = -\frac{t^2\sigma^2}{2} + o(t^2)$.

Hence $|\varphi(t)| = r(t) = \exp\left(-\frac{t^2\sigma^2}{2} + o(t^2)\right)$.

Hence $|\varphi(t)| \leq \exp\left(-\frac{t^2\sigma^2}{4}\right)$ for t sufficiently small

Hence, if $\varepsilon>0$ is chosen sufficiently small, we have

$$I_2 \leq 2\int_A^{\varepsilon\sigma\sqrt{n}} \left(e^{-\left(\frac{t}{\sigma\sqrt{n}}\right)^2 \cdot \frac{\sigma^2}{4}}\right)^n dt \leq 2\int_A^{\infty} e^{-t^2\sigma^2/4}\,dt$$

which is $\leq \varepsilon'$ provided that A is sufficiently large. This completes the proof $\blacksquare$

etc.)

Remark 1: It is possible to prove the following stronger result (useful in particular if $|k|$ is very large):

$$\sup_{k\in\mathbb{Z}} \max\left(1, \left(\frac{k-an}{\sqrt{n}}\right)^2\right) \left| \sigma\sqrt{n}\,\mathbb{P}(Z_n=k) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{k-an}{\sigma\sqrt{n}}\right)^2\right)\right| \xrightarrow[n\to\infty]{} 0$$

See Principles of Random Walk, Chapter 7, by Spitzer

Remark 2: When $\sigma^2=\infty$, it is possible to extend the Local Limit theorem when the law of Z_1 belongs to the domain of attraction of a stable law of index $\alpha\in(0,2)$.

If $\mathbb{P}(Z_1<-1)=0$, this is equivalent to the fact that $\mathbb{P}(Z_1\geq k) = \frac{L(k)}{k^\alpha}$ where L satisfies

$$\frac{L(\lfloor an\rfloor)}{L(n)} \xrightarrow[n\to\infty]{} 1$$

for every $a>0$